1-1-2003

Using data on early returns of electronic products to forecast future availability

Suphalat Chittamvanich
Iowa State University

Using data on early returns of electronic products to forecast future availability

by

Suphalat Chittamvanich

A thesis submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Industrial Engineering

Program of Study Committee:
Sarah Ryan, Major Professor
Mark Kaiser
Jo Min

Iowa State University

Ames, Iowa

2003

Graduate College
Iowa State University

This is to certify that the master's thesis of

Suphalat Chittamvanich

has met the thesis requirements of Iowa State University

Signatures have been redacted for privacy

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1. INTRODUCTION

1.1 Overview

Reverse logistics is defined as "the process of moving goods from their typical final destination to another point, for the purpose of capturing values otherwise unavailable, or for the proper disposal of the product" (The Reverse Logistics Executive Council, 2002). Reverse logistics is considered an environmental friendly practice because it involves activities such as reuse, refurbishment, and recycling while diverting material from landfills (Guide and Van Wassenhove, 2001). Gungor and Gupta (1999) classified these options into two major categories: recycling and remanufacturing. Recycling is "a process of disassembling a product to material level, sorting the materials, and transforming them into a reusable form", while remanufacturing is "an industrial process in which worn-out products are restored to like-new condition. Through a series of industrial processes in a factory environment, a discarded product is disassembled. Only usable parts are cleaned, refurbished, and put into inventory. Then the new product is reassembled from old, and where necessary, new parts to produce a unit fully equivalent – and sometimes superior – in performance and expected lifetime to the original new product" (Lund, 1984). Furthermore, sustainable development is achieved when choosing the reverse logistics as a channel to allow the obsolete product including end-of use and end-of-life to be reused and processed. Goggin and Browne (2000) suggested, "sustainable development promotes the sustaining of resources input to a product so as to gain maximum benefit from these already consumed resources".

Surprisingly, despite acquiring reusable material from recycling, this process also requires a lot of resources so that the small profit it generates makes it not viable. As a result, remanufacturing practice is more desirable because it reduces material procurement cost and consumes fewer natural resources so that it is feasible to generate a significant profit from the recovery process (Ritchy et al, 2001).

Currently, remanufacturing electronics is becoming more prevalent and there are a variety of products involving a recovery process; e.g., single-use cameras, PCs, copy machines etc. Generally, reusable parts that are still working from returns will be reused as input for new products in order to minimize procurement cost for new material. Grenchus et al. (2000) discussed an opportunity to recover value from used PCs. They suggested that some components taken from returned PCs can be used as a parts for other older computers that are still in service due to the fact that near the end of product life, when parts are no longer manufactured, then repair parts are rare and sold at a higher price.

Firms involved in remanufacturing may be either the original equipment manufacturer (OEM) or a third party company that works on value-added recovery (de Brito et al., 2002). The third party company is a specialized company that works on collecting returns or it could be just retailers who collect returns from customer directly. They might either recover valuable material in house or just distribute returns back to OEMs. Generally products are returned because they do not work properly or they do not meet the customer's needs (de Brito et al, 2002).

However, one of the characteristics that complicate the remanufacturing process is the problem of uncertainty in timing and quantity of returns (Guide et al, 2000). More specifically, it can be difficult or impossible to predict which product and how many will be returned in the future. We can find a wide range of examples of this problem in numerous industries; e.g., reusable beverage containers, disposable camera, batteries, toner cartridges, computers, and automobiles. Forecasting returned products in a remanufacturing environment is a process of estimating future product availability by observing past and current sales along with early returns, in order to facilitate the management of remanufacturing operations.

As mentioned earlier, forecasting the returns has been the subject of study for a variety of products; however, the application to electronic goods has been limited and needed to be explored in more detail.

1.2 Electronics recycling

Recently not only the OEMs but also third party recycling companies have faced a new challenge to deal with the enormous amount of returned products from the end users (Lee et al, 2002). This situation happens because of different reasons; e.g., OEMs want to comply with environmental restrictions or want to promote their concern about sustainable development, or just desire to take a benefit from material recovery of used products. Besides, on average the typical return rate for all products in the U.S. is about 6%; while for some leading electronic brands the average return is about 8.46% (Lee et al, 2002). This could be due to rapid advances in technological innovation; recently the useful life of electronic products has been considerably decreased. The National Safety Council (NSC) reported in 1999 that the average lifespan of a personal computer (PC), which was 4.5 years in year 1992 would be reduced to only 2 years in 2005, estimating that more than 315 million PCs would be obsolete by 2004. This report has been confirmed lately by Grenchus et al. (2002), who stated that "the useful life of PC has dropped to between 2 and 3 years".

Dumping outmoded electronic products in a landfill is impractical since the natural resources for building new landfills are becoming exhausted and the returned volume is enormous. In addition, to simply leave an obsolete product in a landfill is not safe to the environment because there is a possibility that a poisonous chemical will leach out into ground water and soil. This environmental impact is the cause of a new environmental mandate that requires a firm located in the European Union to initiate an operation to take back their product after it becomes obsolete.

As mentioned above, material recovery is a desired alternative for used electronic components in order to minimize the environmental impact. Numerous studies have claimed that recoverable manufacturing or remanufacturing can lead to the goal of sustainable development and this activity is also considered as a value-added business process. Guide and Van Wassenhove (2001) suggested that there is a high potential for reuse of products in the consumer electronic market. This reuse operation generates over 53 billion dollars in the total sales per year and is estimated to reduce the manufacturing cost of a new product by 60 to 80% (Guide et al, 2000). There are several ways to recapture asset value from the recovered products:

- Sell via outlet: Several manufacturers have opened outlet stores across the country to sell off returns because this alternative has been proved to provide a better margin than simply selling to a retailer (Roger and Tibben-Lembke, 1999)

- Sell to secondary markets: Firms in this category; e.g., liquidators, wholesalers, etc, sell products at low prices (Roger and Tibben-Lembke, 1999)
- Remanufacture or refurbish: To conserve the product identity by repairing the item into a new condition, especially prevalent in electronic and appliance industries (Fleischmann et al, 1997)
- Auction in the internet: The payment to an internet auctioneer is smaller than the cost of shipping the products back and disposing of them (Richardson, 2001)

Recently several leading electronic companies have initiated a material recovery from product takeback; e.g., Hewlett Packard, Kodak, IBM, Dell etc. HP started its product takeback (PTB) program to allow either business or individual consumers around the world to return used toner-cartridges with no charge (Degher, 2002). Kodak has a worldwide extensive program to reclaim single-use cameras from customers (Degher, 2002). IBM and Dell Corporations have a channel for customers to turn in old PCs on an exchange program.

Even with large quantities of returns, the variability of returns with respect to timing and quantity also impedes the effective management of remanufacturing operations. With this uncertainty additional resources must be available to buffer against an irregular stream, for example, additional space for inventory, labor, and machines; therefore, planning and control in reverse logistics in remanufacturing is difficult (Kokkinaki et al, 2000). Even though there are several factors that affect the estimating of electronic returns, with the leading technology improvement we can forecast the returns from detailed information by keeping track of individual returns. We can identify each item's movement by relying on techniques such as low-cost radio frequency tags (Kokkinaki et al, 2000). Saar and Thomas (2002) discussed the benefit of these tags, which can give the detailed tracking information of recycling products. With the current price of radio frequency tags less than $1 each, it is feasible to use them to track the recycling patterns of various products. When the individual product arrives at the recycling plant then valuable information related to return flows would be obtainable and this information could be applied for a better production planning of recycling process. Brockman (1999) envisioned that warehousing in the 21st century would be likely to have bar coding and radio frequency as powerful tools to record real time data automatically. In a survey conducted on reverse logistics, Rogers and Tibben-Lembke (1999) also found that these tracking devices have already been installed or are planned to assist reverse logistic processes.

1.3 Problem statement

Generally, dealing with returned items is hard because timing and quantity of returns are difficult to predict. To help overcome this problem we can utilize the information from early returns for making decisions about managing remanufacturing activities in a profitable way. This work intends to represent a real situation from the viewpoint of an OEM who tries to manage returns from the market. Mainly an OEM perceives only the time at which each item was sold but does not know when an item will be returned; consequently, it is imperative to utilize data gathered from early returns to determine essential information on potential returns in the future such as the distribution of return times, the mean time that an item will be returned, and the proportion of items that will be returned before they become unprofitable for remanufacturing.

This information not only facilitates planning activities for remanufacturing but it also can be used to determine the viability of remanufacturing for that particular item.

By using the information obtained from tracking the movements of individual items (when each item is sold and returned), we should have the ability to estimate timing and quantity of returns with accuracy. With all mentioned above, this research utilizes the benefit of information from previous sales and earlier returns and applies this information to forecast the future availability of returned items.

1.4 Research objective

The purpose of this thesis is to explore a method of forecasting the returns in an electronic remanufacturing environment. The formulated model assumes that information about each individual item's movement is obtainable; in other words, we assume that information related to when an item was sold and when the same item is returned is available. We investigate how the predictability of future returns changes with variability in sales times and the mean time of return, how prior knowledge of the return time distribution contributes to the precision of estimates, and how these estimates improve as more actual returns arrive.

In this research we propose an idea for estimating necessary parameters for the distribution of the time until electronic goods are returned. A maximum likelihood method for censored data is used repetitively to estimate the parameters as more returns are collected.

The results of this study could be useful for obtaining more insight about the returns and in order to make better decisions about managing operations in remanufacturing environment. Finally, this thesis is especially relevant to the electronic goods, which have a fairly short useful life of approximately between 2 to 4 years, due to rapid turnover of technological advances.

1.5 Thesis organization

The remainder of this thesis is organized into 5 chapters. Chapter 2 reviews the past literature relevant to forecasting product returns. Chapter 3 describes the statistical model for a gamma distribution with censored data and includes the maximum likelihood estimation concept. Chapter 4 shows numerical examples that illustrate the implementation of the model. Finally, Chapter 5 presents concluding remarks and future work.

CHAPTER 2. LITERATURE REVIEW

2.1 Overview

The major frameworks that have been proposed to cope with returned products can be characterized into several groups (Guide et al, 2000). This research will focus on the problem of uncertainty in timing and quantity of returns. Uncertainty in either timing or quantity of returned products impedes good management in procurement decisions, capacity planning and disposal planning (Toktay et al, 2000). Typically the main issue is that the pattern of the return stream is hard to predict and there are several factors that affect the return process, for example computers that are used by business organizations have a useful life of two to three years but conversely individual or household computers might rather last for over 10 years before they might be considered lost, and never to be returned (Grenchus, 2000).

2.2 Returns forecasting

Goh and Varaprasad (1986) determined the reusable-container movement parameters, e.g., the total number of trips made by a container in its lifetime, the average trip duration starting from issuing from the plant and ending upon return to the plant, container life, and the container loss rate. They modeled the returns of reusable containers as proportion of sales volume from present and all earlier returns. Finally they applied the Box and Jenkins statistical approach to estimate those necessary parameters.

Kelle and Silver (1989a, 1989b) developed a forecasting method to estimate the returns of reusable container such as for beverage and liquid gases, based on a variety of available information, e.g., only the proportion of containers returned, the actual issues during each previous period, with records of individuals issued and returned for each period, and the total amount of returns in each previous period without individual identification. They observed the forecast error for each available information case and also suggested that in the case where individual information is known, a substantial improvement of estimating could be achieved; however, this kind of information is very expensive or even impossible to obtain.

Krupp (1992) presented an algorithm to determine the total number of obsolete products that are expected to exist after the end of the product life cycle. His model assumed an environment in which the customer purchases a remanufactured item but the customer may not be required to return the same item for each individual sale.

Srivastava and Guide (1995) proposed two-step approaches to forecast both used product availability and material recovery rates from returns. They suggested that the market growth curve or product life cycle curve can be employed to provide an estimate about used product availability and that the material recovery rate also follows an inverse relationship with used product availability, e.g., as product life-cycle increases, more used products will be available but there is less product recovery due to the wear and tear in products.

Hess and Mayhew (1997) considered the merchandise returns problem and offered both a split adjusted hazard model and a regression model with logit split to estimate the return timing. They explained that the split hazard model, commonly used in the measurement of reliability, uses all of the observations (information from returned and nonreturned), unlike a split regression model that uses only data from returned items, so that a split hazard model can explain not only the timing but also the probability of return. The results showed that the hazard model is more robust and offers a better estimation than the regression model using observations of actual returns from an apparel data set.

Toktay et al. (2000) studied the returns forecasting problem from data on single-use cameras obtained from Kodak. They modeled the return flow with a geometrically distributed lag between sales and return and the data considered as right-censored (more detail about right-censoring will be discussed in the next chapter). A Bayesian approach and the Expectation Maximization (EM) algorithm, a way of doing maximum likelihood estimation, are used to estimate the probability that a product will be returned and the probability that a sold item is returned in the next period given that it will be returned. Two information structures, aggregate and individual tracking data, are considered. In this case, the aggregate data are taken from the volumes of sales and returns in each interval and individual tracking data means the observations are acquired from individual product movements so that we can determine how long each item has spent in the market after being sold. The result showed that the EM algorithm dominates Bayesian estimation even with only a few data available. Finally they suggested that when the demand for remanufacturing is low, using individual tracking data is more favorable; on the other hand, if demand is relatively high, less or only aggregate information is sufficient for the estimation.

In this thesis we extend the idea of tracking individual product movements and apply it to electronic product returns to estimate necessary parameters of future returns with confidence intervals. The predictability is evaluated for different amounts of variability in the sales time and varying expected time to return.

CHAPTER 3. STATISTICAL MODEL

3.1 Introduction

Solomon et al. (2000) described the distribution of electronic product sales over time as following a bell-shaped curve and characterized it into five phases: introduction, growth, maturity, decline, and obsolescence. De Brito et al. (2001) studied the distribution of the returns by analyzing real data. They suggested that the time to return of an item could be modeled by a negative exponential distribution. Toktay et al. (2000) explored the returns from data on single-use cameras obtained from Kodak and modeled the time to return with a discrete time distributed lag model using geometric and pascal distributions. In our model we consider two random variables: (1) time to sale of an individual unit, for example an individual PC, from when the product; e.g., Pentium 4, is introduced to the market, and (2) time from sales to return of each unit. We choose the gamma distribution to represent both random variables. The shape and scale parameter are α and β respectively. Varying the shape parameter allows the density function to take on a variety of shapes. Figure 1 illustrates probability density functions for gamma distribution different values of α but the same β. The scale parameter determines how stretched out the distribution is. The greater the magnitude, the greater the stretching horizontally and compressing in vertically. Figure 2 illustrates probability density functions for the gamma distribution with a constant α and different values of β. The flexibility in the shape of the gamma distribution has made it possible to model a wide variety of distributional shapes unlike other distributions such as normal that has a fixed shape. In addition, the gamma distribution with a large shape parameter can be used to approximate a normal probability density curve. However, unlike a normal random variable, a gamma random variable can take on only nonnegative values, which make it more suitable to model for time intervals. It has a reproductive property, which states that sum of two independent gamma distributed random variables with possibly different shape parameters (α', α'') but with common values of the scale parameter (β) also has a gamma distribution with the same value of β and with $\alpha = \alpha' + \alpha''$ (Johnson et al, 1994, p 340).

We forecast the parameters of the product return distribution using maximum likelihood estimation (MLE) for the gamma distribution with censored observations and extend this estimation framework to determine the confidence intervals for estimated parameters such as the shape and scale parameters, the mean and a cumulative probability. We assume that all items will be returned eventually. The method is illustrated under two scenarios. In the first scenario, we assume that the return time distribution has a known common scale parameter for sales and distribution of time to return. In the second scenario, the method is applied to a situation when information related to the scale parameter could not be acquired.

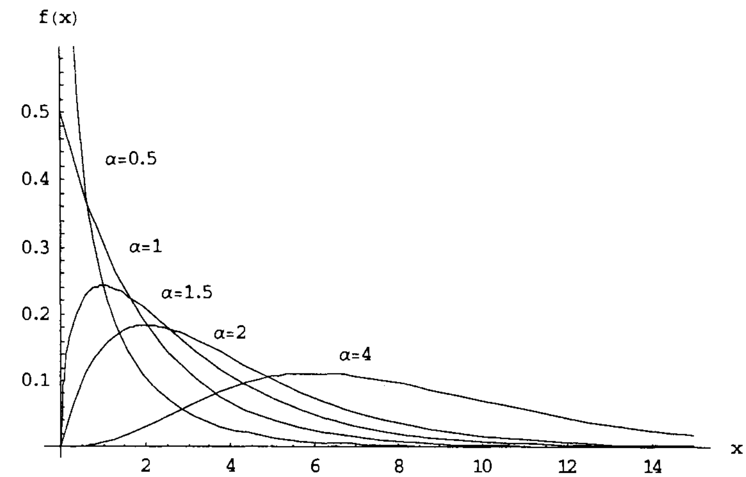


Figure 1. Probability density for gamma distributions with $\alpha = 0.5, 1, 1.5, 2, 4$ and $\beta = 2$.

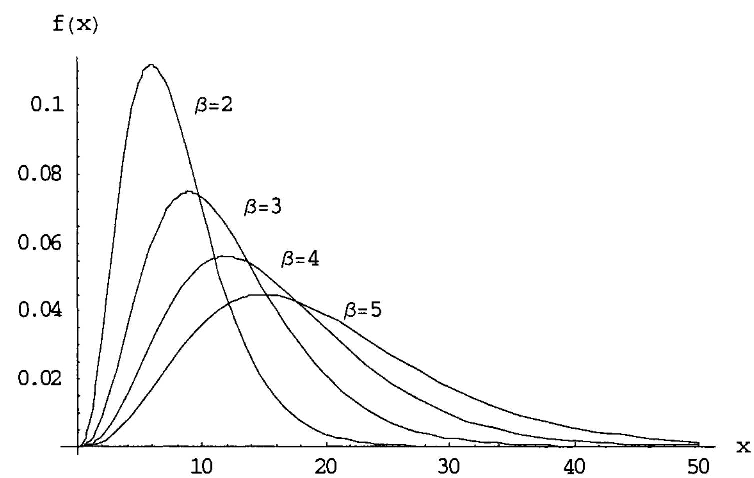


Figure 2. Probability density for gamma distributions with $\alpha = 4$ and $\beta = 2,3,4,5$

3.1.1 Notations

The following describes notations used in this thesis.

i : Index number of products (i =1,2,…., n)

j : Index number of censoring time period

$Y_{1,i}$: Time to sale of unit i from when the product is introduced to the market

$Y_{2,i}$: Time from sale to return of unit i

T_i : Total time until the return of unit i ($T_i = Y_{1,i} + Y_{2,i}$)

S_j : Censoring times

$Y_{ij} = Min(T_i, S_j)$: Observation of T_i at censoring time S_j

D_j : Set of uncensored data at time S_j, $D_j \equiv \{i: T_i \leq S_j\}$

C_j : Set of censored data at time S_j, $C_j \equiv \{i : T_i > S_j\}$

T_0 : Time before the product is worthless starting from product introduction to the market

β : Scale parameter

α_1 : Shape parameter for sales distribution

α_2 : Shape parameter for distribution of time from unit sale to return

$\alpha_3 \equiv \alpha_1 + \alpha_2$

$\hat{\alpha}, \hat{\beta}$: MLE of α, β

$R(\alpha_3, \beta) \equiv P(T_i < T_0)$: Probability that unit i will be returned before time T_0

θ : True value of parameter

$\hat{\theta}$: MLE of θ

3.2 Product return model

3.2.1 Time to sale distribution

We assume that $\{Y_{1,i} : i = 1,...,n\}$ are independent and identically distributed random variables having a common gamma distribution with parameter α_1 and β, which is denoted as $Y_{1,i} \sim iid$ gamma(α_1, β). That is, the probability density function of $Y_{1,i}$, for $i = 1,..., n$, is

$$f(y|\alpha,\beta)=\frac{1}{\beta^{\alpha}\,\Gamma(\alpha\;)}y^{\alpha-1}\exp(\frac{-y}{\beta}),0<y \qquad (1)$$

where Γ (.) is the gamma function defined as,

$$\Gamma(z)=\int_{0}^{\infty}u^{z-1}e^{-u}du \qquad ,0<z \qquad (2)$$

The shape parameter is denoted by α and the scale parameter is denoted by β.

With this parameterization of the gamma distribution, the mean and variance are given as

$$\mu=\alpha\beta$$

$$\sigma^{2}=\alpha\beta^{2}$$

The variability in the distribution may be measured by the coefficient of variation (C.V.) . The C.V. is defined as a ratio of standard deviation to its mean, which is equal to $\frac{1}{\sqrt{\alpha}}$ for gamma distribution. From this we can say that the higher the C.V., the higher the variability relative to location, and the lower the C.V., the higher is the consistency of the data or the lower the variability relative to location. To follow a bell-shaped curve of the sales distribution of electronic products (Solomon et al., 2000), in our model, the time to sales of unit i ($Y_{1,i}$) is assumed to follow a gamma distribution (α_1, β) with a large shape parameter α_1, which makes the curve similar to normal shape.

3.2.2 Time from sales to return

The times from sale to return of unit i $\{Y_{2,i}:i=1,\ldots,n\}$ or the times the units spend with the customer, are *iid* gamma(α_2,β) random variables with the same value of the scale parameter as for $\{Y_{1,i}\}$, which $Y_{2,i}$ is independent of $Y_{1,i}$.

3.2.3 Return time distribution

Let $T_i : i = 1,...,n$ denote the time at which unit i is returned, where time 0 represents product introduction: $T_i = Y_{1,i} + Y_{2,i}, i = 1,...,n$. The reproductive property of the gamma distribution with common parameter β (Johnson et al, 1994, p 340) implies that $\{T_i : i = 1,...,n\}$ are independent and identically distributed gamma random variables with parameters $(\alpha_1 + \alpha_2)$ and β.

We analyze the return data in each time period as incomplete observations. Generally "missing or censored data occurs when some individual data may not be observed for the full time. Therefore only a portion of the individual time is known and the remainder of the time is observed merely to exceed a certain time value" (Cox et al, 1984). Suppose that S, a censoring time, is a period of observation such that observation on the individual ceases at S if its return time has not occurred by then (using notation from Lawless, 1982).

Let Y_{ij} be the observed return time of product i at censoring time S_j

$$Y_{ij} = \text{Min}\ (T_i, S_j) \qquad (3)$$

If $T_i \le S_j$, item i is an uncensored datum and if $T_i > S_j$, item i is a censored datum.

Let $D_j \equiv \{i : T_i \le S_j\}$ be the set of indices for uncensored data and $C_j \equiv \{i : T_i > S_j\}$ be the set of indices for censored data at time S_j.

In our model, usually the exact numbers of the censored items or observations at different censoring times are not known in advance; on the other hand, we can identify just the numbers that are greater than or equal to censoring times. We refer to these observations as Type I censoring at S_j (Lawless, 1982). This type of censoring results in what are also called "right-censored" data, which implies that if the event of interest is to the right of the censoring time then it will be excluded from analysis. Therefore, we have both the set of individuals for whom lifetimes are observed (D_j) and the set of individuals for whom only censoring times are available (C_j).

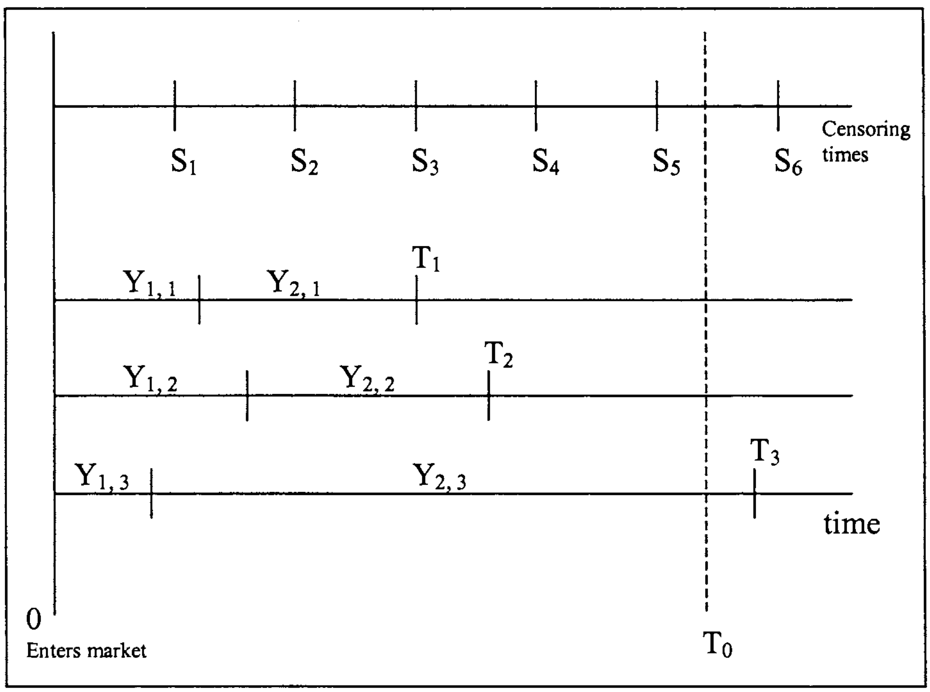


Figure 3. The simulated data and censoring periods

Figure 3 illustrates how we censor observations at different censoring times. At time S_2 all individuals are censored, at time time S_3 only the return time of unit 1 is observed and units 2 and 3 are censored, and at time S_5 units 1,2 have been returned but unit 3 is still censored. In fact, unit 3 will not be returned until after the product is obsolete.

3.2.4 Maximum likelihood function

We estimate the parameters with MLE for the gamma distribution with censored data. This statistical technique is well known and has been extensively used in the field of engineering, reliability, and applied statistics to fit parametric distributions when partial observations are collected. For a particular censoring time, S, we have both sets of observed and censored data.

The likelihood function for a censored sample is:

$$l(\alpha_3,\beta)=\left[\prod_{i\in D}\frac{1}{\beta\Gamma(\alpha_3)}\left(\frac{T_i}{\beta}\right)^{\alpha_3-1}\exp\left(\frac{-T_i}{\beta}\right)\right]\left[\prod_{i\in C}Q\left(\alpha_3,\frac{T_i}{\beta}\right)\right] \qquad (4)$$

where $Q\left(\alpha,\frac{T_i}{\beta}\right)=\frac{1}{\Gamma(\alpha)}\int_{\frac{T_i}{\beta}}^{\infty}u^{\alpha-1}e^{-u}du$.

For computational convenience, it is more common to work with the log-likelihood function instead of likelihood function itself. The logarithm of the likelihood (5) is called the log-likelihood function. For a set of observed product return times such that $|D| = r$ and $|C| = n - r$ the log likelihood may be written in terms of,

$$\bar{t}=\sum_{i\in D}\frac{T_i}{r} \quad \text{and} \quad \tilde{t}=\left(\prod_{i\in D}T_i\right)^{\frac{1}{r}} \quad \text{as}$$

$$L(\alpha_3,\beta)=r\left[(\alpha_3-1)\log(\tilde{t})-\alpha_3\log(\beta)-\log\Gamma(\alpha_3)-\frac{\bar{t}}{\beta}\right]+\sum_{i\in C}\log\left[Q\left\{\alpha_3,\frac{T_i}{\beta}\right\}\right] \qquad (5)$$

where n denotes the total number of products sold, and r denotes the total number of products returned so far.

3.2.5 Parameter estimations

We use MLE to derive the estimators for scale and shape parameters. For our model these values $(\hat{\alpha}, \hat{\beta})$ are not available in analytical form but must be found numerically. We use the *FindMinimum* function in *Mathematica* (Wolfram, 1991, p 1135), which searches for a local minimum. This function employs the Newton-Raphson method to search for local values that maximize the log-likelihood function (Lawless 1982, appendix F).

3.2.6 Interval estimations

The technical conditions necessary for maximum likelihood estimates to be asymptotically normal (AN) (Serfling, 1980) are met for our model (Lawless, 1982, p 525-526). Thus, inference may be based on the fact that

$$(\hat{\alpha}, \hat{\beta}) \text{ is } AN\left((\alpha, \beta), I_{tot}^{-1}(\alpha, \beta)\right) \tag{6}$$

where I_{tot} is the Fisher observed information in a random sample (see Appendix C).

Additionally, we also consider that a returned unit will be worthless after some specific time T_0 after the product introduction. We determine the expected proportion of used items returned before that time period in terms of probability, which we will discuss in more detail later in this chapter.

3.3 Estimating a single parameter

We assume in this case that we can obtain information from the sales data so that the scale parameter (β) is known. The maximum likelihood function in this scenario is used only to estimate the shape parameter. Thus, (5) can be written in the following form.

$$L(\alpha_3) = r\left[(\alpha_3 - 1)\log(\tilde{t}) - \alpha_3 \log(\beta) - \log\Gamma(\alpha_3) - \frac{\bar{t}}{\beta}\right] + \sum_{i \in C} \log\left[Q\left\{\alpha_3, \frac{T_i}{\beta}\right\}\right] \qquad (7)$$

3.3.1 Parameter estimation

Setting the derivative of (7) with respect to α_3 equal to zero and solving for α_3 yields the MLE $\hat{\alpha}_3$. However, we employed the *FindMinimum* to search for local minimum value automatically.

3.3.2 Interval estimation

-Parameters

The Fisher information is merely the observed information from α_3 since we assumed that we know the β value.

Thus, $$I_{tot}(\alpha_3) = -\frac{\partial^2}{\partial \alpha_3^{\,2}} L(\alpha_3) \qquad (8)$$

By the asymptotic normality of MLE (Appendix C), then $\hat{\alpha}_3$ is $AN\left(\alpha_3, I_{tot}^{-1}(\alpha_3)\right)$.

We estimate $V(\alpha_3)$ by $V(\hat{\alpha}_3) = \frac{1}{I_{tot}(\hat{\alpha}_3)}$ so that a $(1-\varphi)100\%$ approximation interval for α_3 is

$$\hat{\alpha}_3 \pm Z_{1-\frac{\varphi}{2}} \left[V(\hat{\alpha}_3)\right]^{1/2} \qquad (9)$$

where $Z_{1-\frac{\varphi}{2}}$ represents a standard normal quartile.

-Mean

We estimate $E(Y_i)=\alpha_3\beta$ by $\hat{\mu}=\hat{\alpha}_3\beta$. By the asymptotic normality and invariance properties of MLE (Appendix C), $g_1(\hat{\alpha}_3)=\hat{\alpha}_3\beta$ is MLE of $g(\alpha_3)=\alpha_3\beta$ and $g'(\hat{\alpha}_3)=\frac{d}{d\hat{\alpha}_3}g(\hat{\alpha}_3)<\infty$ and $g'(\hat{\alpha}_3)\neq 0$.

By the Delta method (Appendix C)

$$g(\hat{\alpha}_3) \text{ is } AN\left(g(\alpha_3),\left[\frac{d}{d\alpha_3}g(\alpha_3)\right]^2\frac{1}{I_{tot}(\alpha_3)}\right) \tag{10}$$

$$\text{so that } \hat{\alpha}_3\beta \text{ is } AN\left(\alpha_3\beta,\frac{\beta^2}{I_{tot}(\alpha_3)}\right). \tag{11}$$

Then a $(1-\varphi)100$ % confidence interval for $\alpha_3\beta$ is

$$\hat{\alpha}_3\beta \pm Z_{1-\frac{\varphi}{2}}\left[\frac{\beta^2}{I_{tot}(\alpha_3)}\right] \tag{12}$$

3.4 Estimating multiple parameters

This case applies to the second scenario in which we do not know the scale parameter but we obtain only times that units are sold and returned from the field so that we must estimate both scale and shape parameters. The log-likelihood function for this scenario is identical to (5).

3.4.1 Parameter estimation:

The MLE of α_3 and β can be obtained by using *FindMinimum* to search for the optimal point of the log-likelihood function.

3.4.2 Interval estimation

-Parameters

The Fisher information for multiple parameters is utilized to find the estimated intervals for parameters and mean. The Fisher observed information is:

$$I_{tot}(\alpha_3,\beta) = -\begin{bmatrix} \frac{\partial^2}{\partial\alpha_3^{\,2}} L(\alpha_3,\beta) & \frac{\partial^2}{\partial\alpha_3\partial\beta} L(\alpha_3,\beta) \\ \frac{\partial^2}{\partial\alpha_3\partial\beta} L(\alpha_3,\beta) & \frac{\partial^2}{\partial\beta^2} L(\alpha_3,\beta) \end{bmatrix} \tag{13}$$

Corresponding with the asymptotic normality of MLE

$(\hat{\alpha}_3,\hat{\beta})$ is $AN((\alpha_3,\beta), I_{tot}^{-1}(\alpha_3,\beta))$

or $AN((\alpha_3,\beta), V(\alpha_3,\beta))$, $V(\alpha_3,\beta) = I_{tot}^{\;-1}(\alpha_3,\beta)$

Let $I_{tot}^{\;-1}(\alpha_3,\beta) = \begin{bmatrix} i^{1,1}(\alpha_3,\beta) & i^{1,2}(\alpha_3,\beta) \\ i^{1,2}(\alpha_3,\beta) & i^{2,2}(\alpha_3,\beta) \end{bmatrix}$. (14)

We estimate it as $I_{tot}^{\;-1}(\hat{\alpha}_3,\hat{\beta}) = \begin{bmatrix} i^{1,1}(\hat{\alpha}_3,\hat{\beta}) & i^{1,2}(\hat{\alpha}_3,\hat{\beta}) \\ i^{1,2}(\hat{\alpha}_3,\hat{\beta}) & i^{2,2}(\hat{\alpha}_3,\hat{\beta}) \end{bmatrix}$ (15)

so the approximate $(1-\varphi)100\%$ confidence intervals for α_3,β are:

$$\hat{\alpha}_3 \pm z_{1-\frac{\varphi}{2}}\left[i^{1,1}(\hat{\alpha}_3,\hat{\beta})\right]^{1/2} \tag{16}$$

$$\hat{\beta} \pm z_{1-\frac{\varphi}{2}}\left[i^{2,2}(\hat{\alpha}_3,\hat{\beta})\right]^{1/2} \tag{17}$$

-Mean

We estimate E(Y$_i$) = $\alpha_3\beta$ by $\hat{\mu} = \hat{\alpha}_3\hat{\beta}$. By invariance of MLE, $g_2(\hat{\alpha}_3, \hat{\beta})$ is MLE for

$g(\alpha_3, \beta) = \alpha_3\beta$

so that

$$E(Y_i) = g(\alpha_3, \beta).$$

Let

$$D = \begin{bmatrix} \frac{\partial g_2}{\partial \alpha_3} & \frac{\partial g_2}{\partial \beta} \end{bmatrix} \tag{18}$$

and by the Delta method

$$g(\hat{\alpha}_3, \hat{\beta}) \text{ or } \hat{\mu} \text{ is } AN\left(g(\alpha_3, \beta), DI_{tot}^{-1}(\alpha_3, \beta)D^T\right)$$

where $DI_{tot}^{-1}(\alpha_3, \beta)D^T$ can be written as

$$[\beta \quad \alpha_3] \begin{bmatrix} i^{1,1}(\hat{\alpha}_3, \hat{\beta}) & i^{1,2}(\hat{\alpha}_3, \hat{\beta}) \\ i^{1,2}(\hat{\alpha}_3, \hat{\beta}) & i^{2,2}(\hat{\alpha}_3, \hat{\beta}) \end{bmatrix} \begin{bmatrix} \beta \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} V(\hat{\mu}) & \text{cov}(\hat{\mu}, \hat{v}) \\ \text{cov}(\hat{\mu}, \hat{v}) & V(\hat{v}) \end{bmatrix}$$

where $V(\hat{\mu})$ represents the variance of $\hat{\mu}$ (mean)

$V(\hat{v})$ represents the variance of $\hat{v}$ (variance)

$\text{cov}(\hat{\mu}, \hat{v})$ represents the covariance of $\hat{\mu}$ and $\hat{v}$.

Hence a $(1-\varphi)$ 100 % confidence interval for $\hat{\alpha}\hat{\beta}$ is

$$\hat{\alpha}\hat{\beta} \pm Z_{1-\frac{\varphi}{2}}[V(\hat{\mu})] \tag{19}$$

3.5 Probability estimations

As mentioned earlier, it is imperative to estimate the proportion of units that will be returned before they are not economical to reuse again. Applying probability concepts we can estimate this quantity as a probability that unit i will be returned before time T_0 or $P(T_i < T_0)$. Thus

$$R(\alpha_3, \beta) \equiv P(T_i < T_0) = \int_0^{T_0} \frac{1}{\beta^{\alpha_3}\Gamma(\alpha_3)} t^{\alpha_3 - 1} e^{-t/\beta} dt. \tag{20}$$

We estimate $R(\alpha_3, \beta)$ with $\hat{R}$ so, $\hat{R} = \int_0^{T_0} \frac{1}{\beta^{\hat{\alpha}_3}\Gamma(\hat{\alpha}_3)} t^{\hat{\alpha}_3 - 1} e^{-t/\beta} dt.$ (21)

By the invariance property,

$\hat{R}(\hat{\alpha}_3, \beta)$ is MLE for $R(\alpha_3, \beta)$.

3.5.1 Estimating a single parameter

Since $\hat{R}(\hat{\alpha}_3, \beta)$ is $AN\left(R(\alpha_3, \beta), \left[\left(\frac{\partial R}{\partial \alpha_3} \right)\bigg|_{\alpha_3 = \hat{\alpha}_3} \right]^2 \frac{1}{I_{tot}(\alpha_3)} \right)$ (22)

then a $(1-\varphi)100$ % confidence interval for $\hat{R}$ is

$$\hat{R}(\hat{\alpha}_3, \beta) \pm Z_{1-\frac{\varphi}{2}} \left[\frac{\left[\left(\frac{\partial R}{\partial \alpha_3} \right)\bigg|_{\alpha_3 = \hat{\alpha}_3} \right]^2}{I_{tot}(\alpha_3)} \right]^{\frac{1}{2}} \tag{23}$$

where

$$\frac{\partial R}{\partial \alpha_3} = \frac{1}{\beta^{\alpha_3}\Gamma(\alpha_3)}\left[\int_0^{T_0} t^{\alpha_3-1}\log(t)e^{-t/\beta}dt - \log(\beta)\int_0^{T_0} t^{\alpha_3-1}e^{-t/\beta}dt - \frac{\Gamma'(\alpha_3)}{\Gamma(\alpha_3)}\int_0^{T_0} t^{\alpha_3-1}e^{-t/\beta}dt\right].$$

3.5.2 Estimating multiple parameters

Let $G = \left[\left(\frac{\partial R}{\partial \alpha_3}\right)\Bigg|_{\substack{\alpha_3=\hat{\alpha}_3 \\ \beta=\hat{\beta}}} \left(\frac{\partial R}{\partial \beta}\right)\Bigg|_{\substack{\alpha_3=\hat{\alpha}_3 \\ \beta=\hat{\beta}}}\right]$

where

$$\frac{\partial R}{\partial \beta} = \frac{1}{\beta^{\alpha_3+2}\Gamma(\alpha_3)}\int_0^{T_0} t^{\alpha_3}e^{-t/\beta}dt - \frac{\alpha_3}{\beta^{\alpha_3-1}\Gamma(\alpha_3)}\int_0^{T_0} t^{\alpha_3-1}e^{-t/\beta}dt .$$

Corresponding to the Delta method,

$\hat{R}(\hat{\alpha}_3, \hat{\beta})$ is $AN\left(\hat{R}(\alpha_3, \beta), GI_{tot}^{-1}G^T\right)$

where $V(\hat{R}) \equiv GI_{tot}^{-1}G^T$.

Hence, a $(1-\varphi)100\%$ confidence interval for $\hat{R}$ is

$$\hat{R} \pm z_{1-\frac{\varphi}{2}}\left[V(\hat{R})\right]^{1/2}. \qquad (24)$$

CHAPTER 4. NUMERICAL EXAMPLES

In this chapter, the MLE presented in the previous chapter are demonstrated and the performance of our point estimators is evaluated. There are two scenarios depending on the prior knowledge of the return time distribution. Each scenario consists of six different cases that describe the sale pattern of different types of products; e.g., high or low variability, and different sources of returns; e.g., corporation or household use. We simulated distributions according to the useful life of PC (Grenchus, 2002).

This work intends to explore the effects of: (1) different patterns of sale distributions, (2) mean time to return from different return origins, and (3) the prior knowledge of the scale parameter on predictability. We measure predictability in terms of the accuracy and precision.

4.1 Factors

4.1.1 Sales distribution

We applied C.V.s of 0.5, 0.35, and 0.25 in the time to sale distribution. Assuming the expected time until a unit is sold is 100, the scale parameters are chosen to equal 25,12.5, and 6.25 correspondingly. The various intensities of variability could represent several types of products sales: Low variability means a product that is slow to gain popularity at the introduction, then sells fast, but the acceptance diminishes shortly after peak in sales. High variability represents a product that is successful in sales volume soon after it is introduced to the market and its sales continue at a steady high for a long time before decreasing.

Table 1. The parameters for time to sale distribution

C.V.	α_1	β
0.500	4	25
0.354	8	12.5
0.250	16	6.25

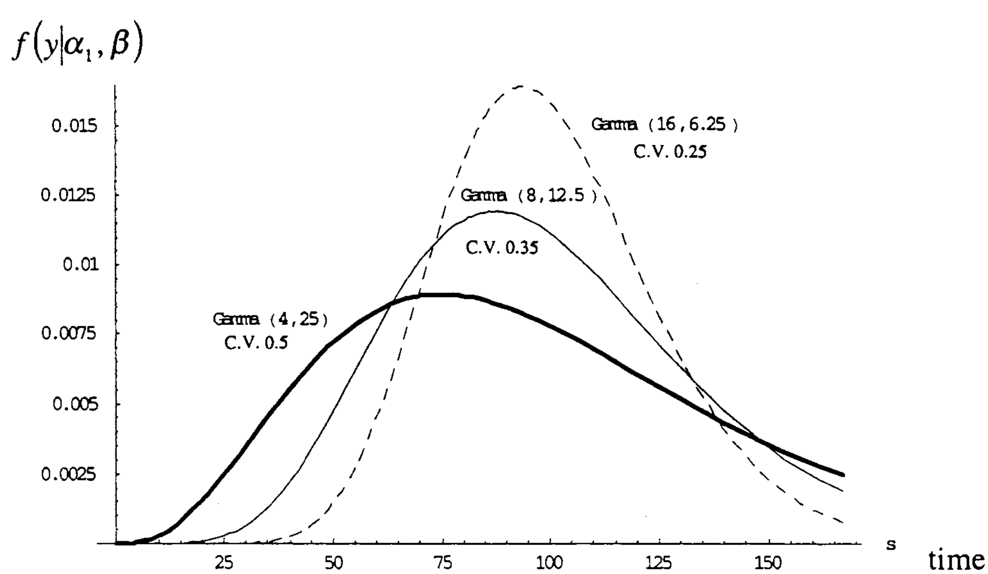


Figure 4. Probability density functions of time to sale with different C.V. values

4.1.2 Time from sales to return

For each $Y_{2,i}$ distribution we held scale parameters constant and selected shape parameters for two different mean times to return according to the estimation that the useful life of a PC is approximately 100 weeks for corporation and about 200 weeks for consumer use (Grenchus, 2000).

Table 2. The parameters for the distribution of the time from sales to return

α_2	β	Mean
4	25	100
8	12.5	100
16	6.25	100
8	25	200
16	12.5	200
32	6.25	200

4.1.3 Lifetime distribution

By applying different variability in sales data and the two alternatives for mean time to return, we can distinguish into 6 cases of lifetime data.

Table 3. The parameters for lifetime distribution

Case	α_1	C.V. of $Y_{1,i}$	μ_1	α_2	C.V. of $Y_{2,i}$	μ_2	α_3	C.V. of T_i	μ_3	β	$R(\alpha_3,\beta)$
1	4	0.500	100	4	0.500	100	8	0.354	200	25	0.981
2	8	0.354	100	8	0.354	100	16	0.250	200	12.5	0.998
3	16	0.250	100	16	0.250	100	32	0.177	200	6.25	0.999
4	4	0.500	100	8	0.354	200	12	0.289	300	25	0.815
5	8	0.354	100	16	0.250	200	24	0.204	300	12.5	0.885
6	16	0.250	100	32	0.177	200	48	0.144	300	6.25	0.950

These six cases were used to study the effect of variability in lifetime data on predictability of returns as more actual returns arrive. As we mentioned earlier, the first three cases are attributed to products returned originally from corporation uses and last three cases simulate returned items from consumer use. A simulation model was developed in order to generate data sets corresponding to different cases and estimate parameters by MLE for censored data. The simulations were conducted in the *Mathematica* program (Wolfram, 1991). We used 100 replicates (each replicate represents 750 products (n = 750)) in the simulation. The censoring times were chosen to be times 75,100,125…600, and 700 (j =1,….., 21). We assumed that an item is worthless if returned after period 375 ($T_0 = 375$).

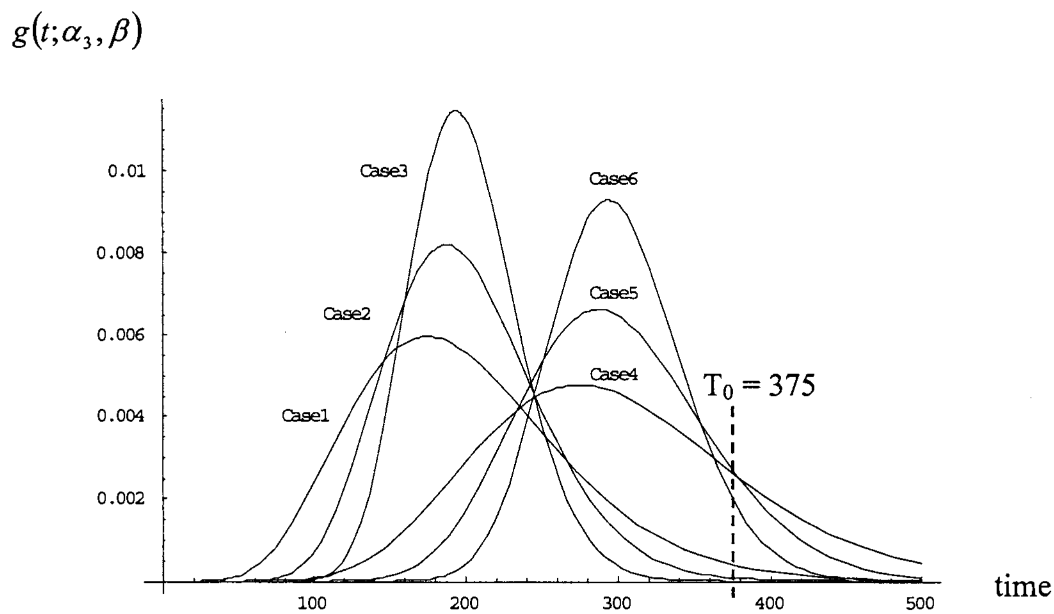


Figure 5. Probability density functions of lifetime distribution

The results represent the average over 100 replicates in each case. Note that $\hat{R}$ and $V(\hat{R})$ are shown only for censoring times that are less than $T_0 = 375$ and we record the true value of $\hat{R}$ censoring after time 350 as the proportion of returns at time 375 to the number of products that were sold (N = 750). Full numerical details for scenarios 1 and 2 including 90% confidence intervals are shown in Appendix A and B.

In order to evaluate the effects of: (1) variability in the distribution of time to sales, (2) expected time to return, and (3) the prior knowledge about return characteristics on predictability based on early returns, we examine our estimates in terms of accuracy and precision by using the average over 100 replicates in each case as an estimate. To emphasize early returns, we consider censoring times only up to time 300. With different mean time to return, cases 1-3 have different starting points from cases 4-6; therefore we will consider early returns from cases 1-3 at times 125-200 and cases 4-6 at times 200-300. Note that each case has a different starting censoring time, for example, the first estimates for case 2 are available at S_2 =100 because before this starting time a lot of items are still missing (more than 99 % censored); as a result, the censoring times with so much missing data are not considered, which we indicate with not available (n/a) in the tables.

4.2 Accuracy of estimation

Accuracy describes the closeness of the estimate to the true value. We measure the accuracy by the deviation from the true value defined as

$$\% \text{ error } = \frac{|\hat{\theta} - \theta|}{\theta} \times 100\% ,$$

such that θ is a true value, and $\hat{\theta}$ is an estimated value.

Scenario 1. Assuming prior knowledge of β is available

Tables 4-6 and Figures 6-8 illustrate the percent deviations from the true values for $\hat{\alpha}_3$, $\hat{\mu}$, and $\hat{R}$ at different censoring times from the scenario 1 data. Note that in this case the percent deviations of $\hat{\mu}$ have the same values as the percent deviations of $\hat{\alpha}_3$ because we fix the β values. The results show that in the long run percent errors in $\hat{\alpha}_3, \hat{\mu}$, and $\hat{R}$ are considerably low with little fluctuation. Cases 6 has higher error in $\hat{\alpha}_3$ and $\hat{\mu}$ from the first censoring time but this large error is due to only a few observations, which could occur in other cases as well as we extend to earlier censoring times. The % errors in $\hat{R}$ are somewhat more steady in cases 1-3 than in cases 4-6.

Table 4. Comparing % error in estimating $\hat{\alpha}_3$ for scenario 1

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	4.795E-01	n/a	n/a	n/a	n/a	n/a
100	4.065E-01	1.593E+00	n/a	n/a	n/a	n/a
125	2.331E-01	3.488E-01	5.384E-01	5.350E-01	n/a	n/a
150	2.992E-01	1.525E-01	5.106E-01	5.250E-01	n/a	n/a
175	3.855E-01	2.194E-01	3.609E-01	6.892E-01	8.067E-01	n/a
200	3.071E-01	1.087E-01	3.378E-01	3.033E-01	7.167E-02	2.636E+00
225	2.223E-01	5.313E-02	3.059E-01	5.025E-01	2.917E-03	7.152E-01
250	2.151E-01	6.187E-02	2.775E-01	2.650E-01	2.129E-01	3.333E-01
275	2.419E-01	9.188E-02	2.684E-01	2.750E-01	1.367E-01	2.202E-01
300	2.166E-01	6.187E-02	2.753E-01	2.592E-01	3.625E-02	1.237E-01

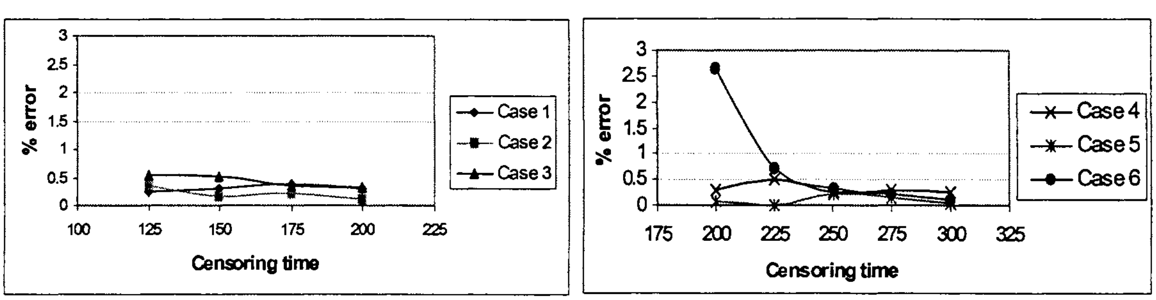


Figure 6. Comparisons of % error in estimating $\hat{\alpha}_3$ with different censoring times for scenario 1

Table 5. Comparing % error in estimating $\hat{\mu}$ for scenario 1

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	4.795E-01	n/a	n/a	n/a	n/a	n/a
100	4.065E-01	1.593E+00	n/a	n/a	n/a	n/a
125	2.331E-01	3.487E-01	5.384E-01	5.350E-01	n/a	n/a
150	2.993E-01	1.525E-01	5.106E-01	5.250E-01	n/a	n/a
175	3.855E-01	2.194E-01	3.609E-01	6.892E-01	8.067E-01	n/a
200	3.071E-01	1.088E-01	3.378E-01	3.033E-01	7.167E-02	2.636E+00
225	2.223E-01	5.312E-02	3.059E-01	5.025E-01	2.917E-03	7.152E-01
250	2.151E-01	6.188E-02	2.775E-01	2.650E-01	2.129E-01	3.333E-01
275	2.419E-01	9.188E-02	2.684E-01	2.750E-01	1.367E-01	2.202E-01
300	2.166E-01	6.188E-02	2.753E-01	2.592E-01	3.625E-02	1.237E-01

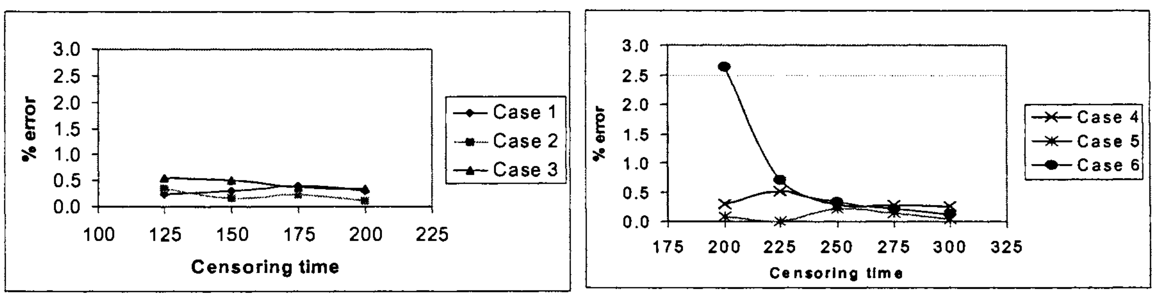


Figure 7. Comparisons of % error in estimating $\hat{\mu}$ with different censoring times for scenario 1

Table 6. Comparing % error in estimating $\hat{R}$ for scenario 1

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	8.747E-02	n/a	n/a	n/a	n/a	n/a
100	6.069E-02	4.709E-02	n/a	n/a	n/a	n/a
125	3.554E-02	7.314E-03	2.000E-04	7.726E-01	n/a	n/a
150	4.623E-02	1.002E-03	2.000E-04	7.159E-01	n/a	n/a
175	5.346E-02	7.014E-04	1.000E-04	8.434E-01	3.230E-01	n/a
200	4.257E-02	4.509E-03	1.000E-04	3.615E-01	5.332E-01	7.246E-01
225	3.157E-02	3.807E-03	1.000E-04	5.887E-01	1.436E-01	1.614E-01
250	2.963E-02	2.906E-03	1.000E-04	5.955E-01	4.315E-02	9.939E-02
275	3.259E-02	6.012E-04	1.000E-04	3.241E-01	2.790E-02	1.452E-01
300	2.963E-02	1.904E-03	1.000E-04	3.071E-01	1.668E-01	1.619E-01

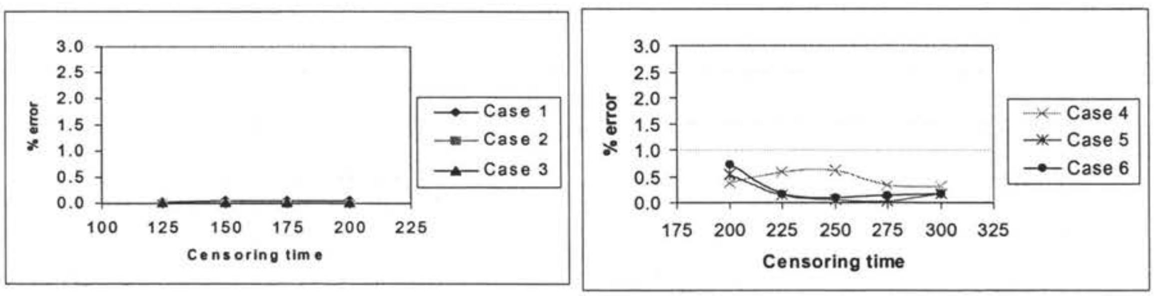


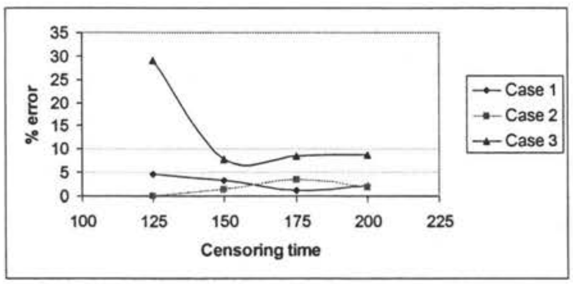
Figure 8. Comparisons of % error in estimating $\hat{R}$ with different censoring times for scenario 1

Scenario 2. Assuming without prior knowledge of β

Tables 7-10 and Figures 9-12 illustrate the percent deviations from the true values for $\hat{\alpha}_3$, $\hat{\beta}$, $\hat{\mu}$, and $\hat{R}$ at different censoring times from the scenario 2 data. The results show that percent errors in every estimate tend to stay steady over time except case 3, which stays around 10 % from the beginning and decreases to 6% at time 300. The errors in $\hat{\mu}$ and $\hat{R}$ are considerably smaller compared to errors in $\hat{\alpha}_3$ and $\hat{\beta}$.

Table 7. Comparing % error in estimating $\hat{\alpha}_3$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	7.996E+00	n/a	n/a	n/a	n/a	n/a
100	1.953E+00	2.811E+01	n/a	n/a	n/a	n/a
125	4.629E+00	2.500E-03	2.894E+01	2.221E+01	n/a	n/a
150	3.251E+00	1.289E+00	7.685E+00	1.891E+01	n/a	n/a
175	1.043E+00	3.324E+00	8.448E+00	1.573E+01	1.245E+01	n/a
200	1.982E+00	1.763E+00	8.793E+00	8.498E+00	8.551E+00	2.099E+00
225	2.462E+00	2.387E+00	9.030E+00	7.270E+00	3.609E+00	7.960E-01
250	2.506E+00	2.158E+00	9.169E+00	1.698E+00	1.859E+00	4.717E-01
275	3.395E+00	2.220E+00	9.199E+00	1.290E+00	1.917E-01	2.860E-01
300	2.765E+00	1.834E+00	6.219E+00	3.333E-02	1.496E-01	1.475E-01



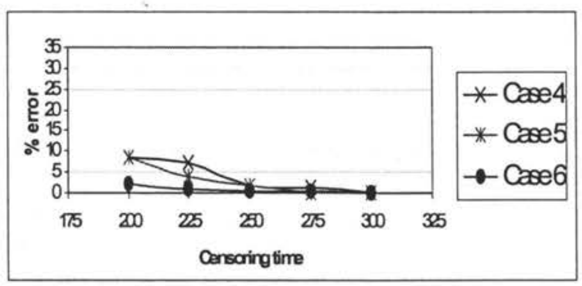


Figure 9. Comparisons of % error in estimating $\hat{\alpha}_3$ with different censoring times for scenario 2

Table 8. Comparing % error in estimating $\hat{\beta}$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	2.155E+01	n/a	n/a	n/a	n/a	n/a
100	1.138E+01	8.088E+01	n/a	n/a	n/a	n/a
125	1.282E+01	2.963E+00	1.003E+02	1.254E+01	n/a	n/a
150	7.328E+00	2.402E+00	1.118E+01	1.138E+01	n/a	n/a
175	4.828E+00	4.287E+00	1.135E+01	1.305E+01	3.823E+01	n/a
200	2.848E-01	1.973E+00	1.134E+01	7.868E+00	2.275E+01	3.252E+00
225	2.336E-01	2.472E+00	1.132E+01	7.034E+00	8.612E+00	1.335E+00
250	1.619E+00	2.255E+00	1.152E+01	1.840E+00	3.090E+00	1.049E+00
275	2.624E+00	2.289E+00	1.017E+01	1.321E+00	1.237E+00	7.166E-01
300	1.957E+00	1.882E+00	7.980E+00	2.504E-01	1.075E+00	5.958E-01

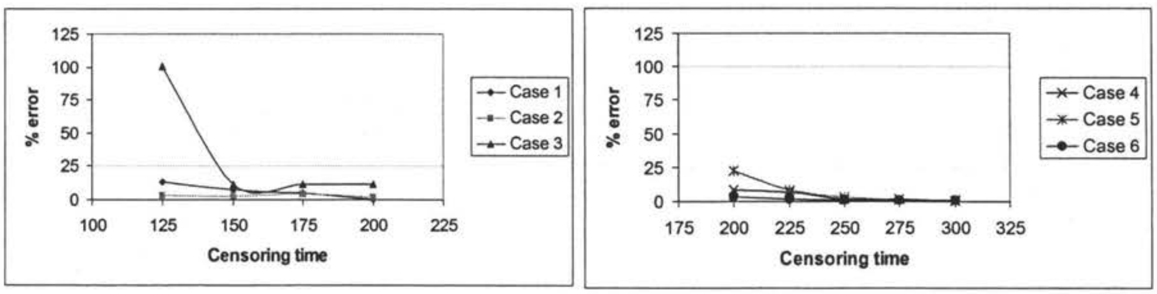


Figure 10. Comparisons of % error in estimating $\hat{\beta}$ with different censoring times for scenario 2

Table 9. Comparing % error in estimating $\hat{\mu}$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	3.127E+01	n/a	n/a	n/a	n/a	n/a
100	9.206E+00	3.004E+01	n/a	n/a	n/a	n/a
125	7.599E+00	2.961E+00	4.229E+01	6.882E+00	n/a	n/a
150	3.839E+00	3.722E+00	2.633E+00	5.378E+00	n/a	n/a
175	3.735E+00	8.210E-01	1.944E+00	6.342E-01	2.103E+01	n/a
200	1.692E+00	2.451E-01	1.546E+00	3.875E-02	1.226E+01	1.085E+00
225	2.223E+00	2.612E-02	1.267E+00	2.758E-01	4.692E+00	5.279E-01
250	8.469E-01	4.904E-02	1.296E+00	1.733E-01	1.173E+00	5.723E-01
275	6.827E-01	1.799E-02	3.536E-02	4.824E-02	1.043E+00	4.285E-01
300	7.539E-01	1.271E-02	1.264E+00	2.838E-01	9.240E-01	4.474E-01

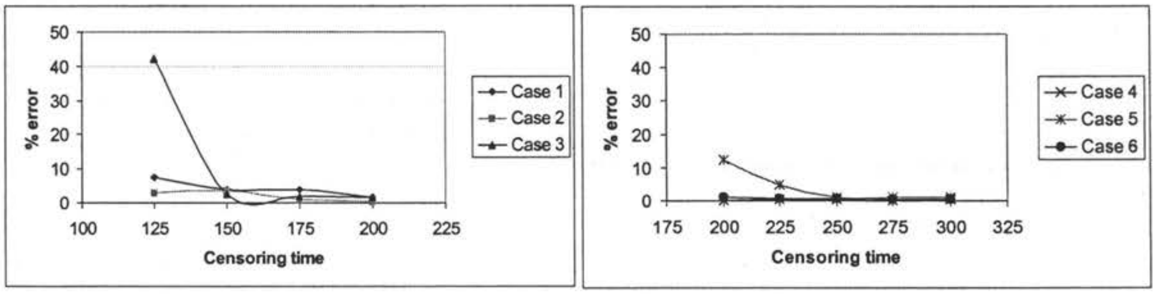


Figure 11. Comparisons of % error in estimating $\hat{\mu}$ with different censoring times for scenario 2

Table 10. Comparing % error in estimating $\hat{R}$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	4.742E+00	n/a	n/a	n/a	n/a	n/a
100	1.504E+00	6.969E+00	n/a	n/a	n/a	n/a
125	1.532E+00	1.724E-01	4.297E+00	4.202E+00	n/a	n/a
150	7.436E-01	2.488E-01	9.600E-03	4.450E+00	n/a	n/a
175	5.270E-01	6.032E-02	7.500E-03	5.220E+00	1.434E+01	n/a
200	1.092E-01	2.395E-02	6.400E-03	3.136E+00	7.044E+00	1.113E+00
225	3.971E-03	2.224E-02	5.600E-03	2.891E+00	2.397E+00	5.039E-01
250	1.120E-02	2.054E-02	5.300E-03	9.342E-01	6.273E-01	5.132E-01
275	7.240E-02	1.974E-02	5.200E-03	8.093E-01	4.344E-01	3.754E-01
300	2.974E-02	1.603E-02	3.700E-03	3.430E-01	3.104E-01	3.794E-01

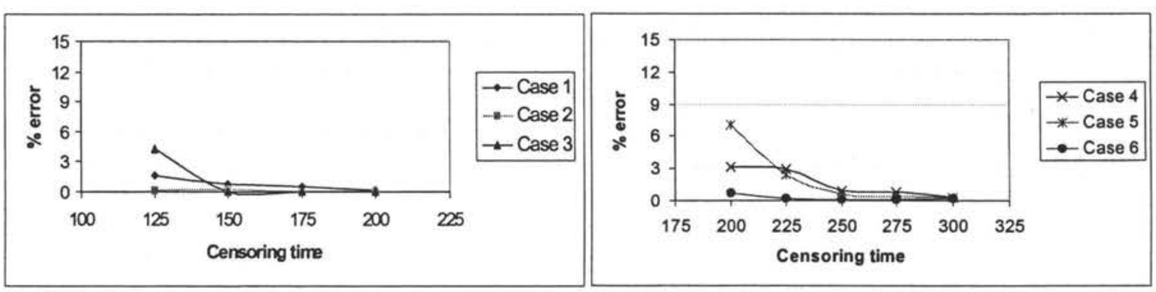


Figure 12. Comparisons of % error in estimating $\hat{R}$ with different censoring times for scenario 2

4.3 Precision of estimation

The precision can be described as the narrowness of the approximate confidence interval width. Precision can be quantified by $\frac{\sqrt{V(\hat{\theta})}}{\hat{\theta}} \times 100$, where $V(\hat{\theta})$ denotes the variance of an estimate. We call this quantity the relative standard deviation (RSD) (National institute of standards and technology, 2003).

Scenario 1. Assuming prior knowledge of β is available

Tables 11-13 and Figures 13-15 illustrate percent RSDs for $\hat{\alpha}_3$, $\hat{\mu}$, and $\hat{R}$ at different censoring times for scenario 1. The findings show that every case has % RSDs decreasing in the censoring time. In addition we can see that for the cases with higher variability of sale distributions, the RSDs will converge to higher value in RSDs than the cases with lower variability. However, RSDs in $\hat{R}$ behave differently and can be combined into two groups, lower or higher mean time to return. The group with lower mean time to return has smaller RSDs than the group with higher mean time to return. In addition, for both groups, the case with higher C.V. also has higher RSDs than lower C.V. at the same censoring times.

Table 11. Comparing % RSD in estimating $\hat{\alpha}_3$ for scenario 1

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	3.883E+00	n/a	n/a	n/a	n/a	n/a
100	2.359E+00	3.569E+00	n/a	n/a	n/a	n/a
125	1.788E+00	1.779E+00	2.418E+00	4.488E+00	n/a	n/a
150	1.531E+00	1.261E+00	1.138E+00	2.745E+00	n/a	n/a
175	1.399E+00	1.052E+00	8.049E-01	1.976E+00	2.735E+00	n/a
200	1.329E+00	9.620E-01	6.927E-01	1.569E+00	1.598E+00	2.258E+00
225	1.291E+00	9.253E-01	6.558E-01	1.358E+00	1.173E+00	1.113E+00
250	1.271E+00	9.087E-01	6.448E-01	1.233E+00	9.623E-01	7.762E-01
275	1.261E+00	9.017E-01	6.420E-01	1.150E+00	8.547E-01	6.293E-01
300	1.255E+00	8.994E-01	6.414E-01	1.103E+00	7.958E-01	5.661E-01

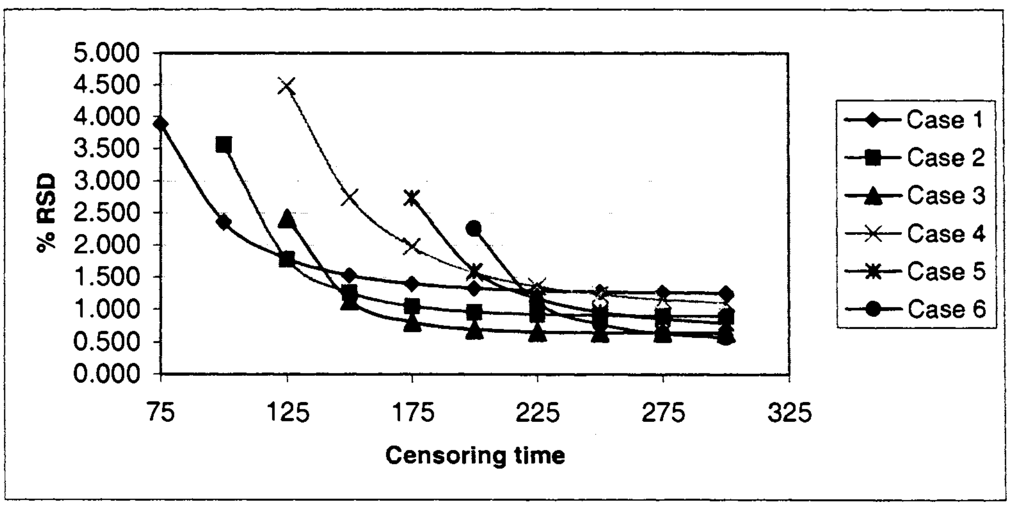


Figure 13. Comparisons of % RSD in estimating $\hat{\alpha}_3$ with different censoring times for scenario 1

Table 12. Comparing % RSD in estimating $\hat{\mu}$ for scenario 1

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	3.883E+00	n/a	n/a	n/a	n/a	n/a
100	2.359E+00	3.569E+00	n/a	n/a	n/a	n/a
125	1.788E+00	1.779E+00	2.418E+00	4.488E+00	n/a	n/a
150	1.531E+00	1.261E+00	1.138E+00	2.745E+00	n/a	n/a
175	1.399E+00	1.052E+00	8.049E-01	1.976E+00	2.735E+00	n/a
200	1.329E+00	9.620E-01	6.927E-01	1.569E+00	1.598E+00	2.258E+00
225	1.291E+00	9.253E-01	6.558E-01	1.358E+00	1.173E+00	1.113E+00
250	1.271E+00	9.087E-01	6.448E-01	1.233E+00	9.623E-01	7.762E-01
275	1.261E+00	9.017E-01	6.420E-01	1.150E+00	8.547E-01	6.293E-01
300	1.255E+00	9.008E-01	6.414E-01	1.103E+00	7.958E-01	5.661E-01

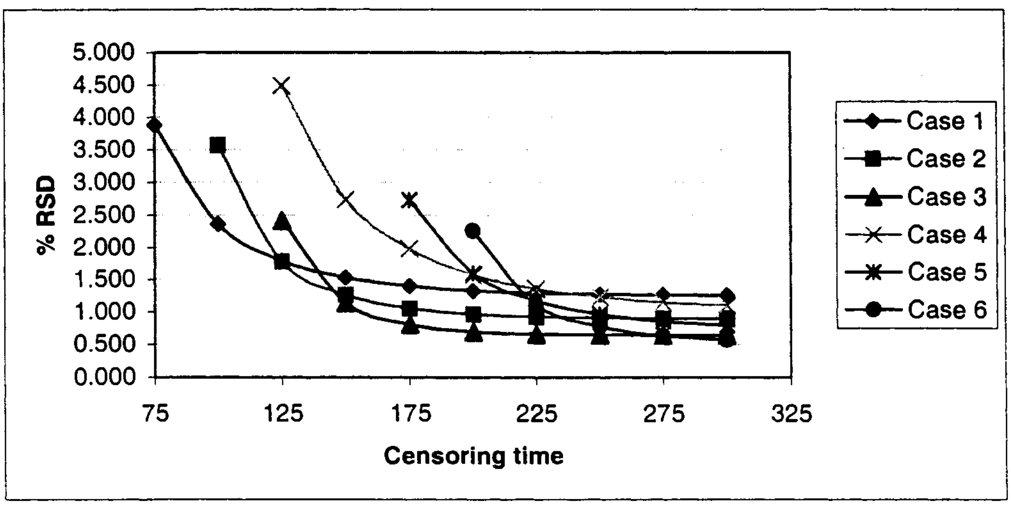


Figure 14. Comparisons of % RSD in estimating $\hat{\mu}$ with different censoring times for scenario 1

Table 13. Comparing % FSD in estimating $\hat{R}$ for scenario 1

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	4.913E-01	n/a	n/a	n/a	n/a	n/a
100	2.868E-01	1.121E-01	n/a	n/a	n/a	n/a
125	2.131E-01	4.228E-02	2.260E-03	5.242E+00	n/a	n/a
150	1.837E-01	2.864E-02	6.595E-04	3.169E+00	n/a	n/a
175	1.680E-01	2.387E-02	4.727E-04	2.276E+00	3.752E+00	n/a
200	1.584E-01	2.225E-02	4.050E-04	1.762E+00	1.644E+00	1.434E+00
225	1.529E-01	2.123E-02	3.838E-04	1.539E+00	1.249E+00	8.446E-01
250	1.503E-01	2.075E-02	3.795E-04	1.393E+00	1.000E+00	5.652E-01
275	1.493E-01	2.036E-02	3.780E-04	1.288E+00	8.876E-01	4.722E-01
300	1.484E-01	2.044E-02	3.773E-04	1.234E+00	8.402E-01	4.207E-01

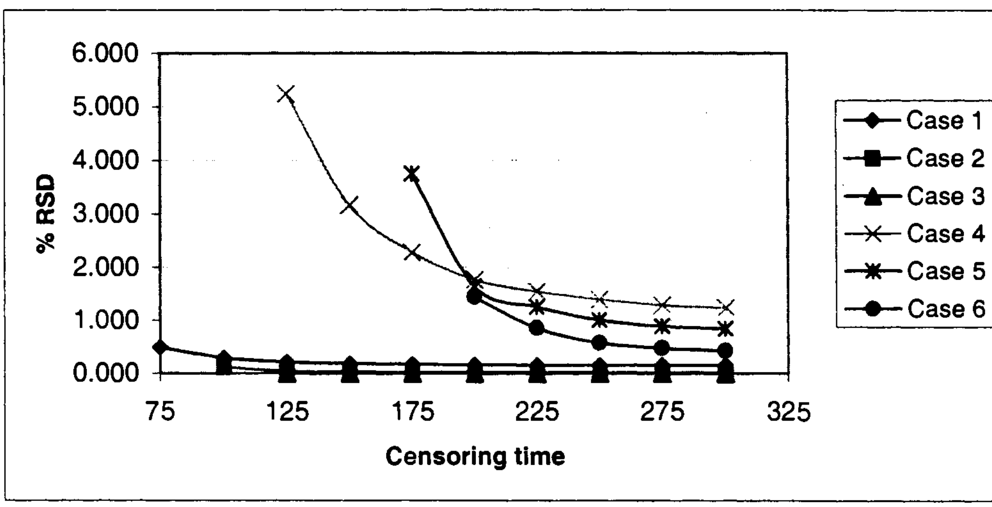


Figure 15. Comparisons of % RSD in estimating $\hat{R}$ with different censoring times for scenario 1

Scenario 2. Assuming without prior knowledge of β

Tables 14-17 and Figures 16-19 illustrate RSDs for $\hat{\alpha}_3, \hat{\beta}, \hat{\mu}$, and $\hat{R}$ at different censoring times for scenario 2. The results show that RSDs tend to decrease and stay steady over time for every case. Generally RSDs for cases 1-3 stabilize faster with lower RSDs than cases 4-6. One possible explanation is that cases 1-3 have lower mean time to return so that the estimation is improved as more returns arrive earlier. Furthermore, RSDs in $\hat{\mu}$ and $\hat{R}$ are lower than those for $\hat{\alpha}_3$ and $\hat{\beta}$, which means that we can obtain narrower confidence intervals for estimating $\hat{\mu}$ and $\hat{R}$ than for the parameter estimates.

Table 14. Comparing % RSD in estimating $\hat{\alpha}_3$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	4.664E+01	n/a	n/a	n/a	n/a	n/a
100	2.435E+01	4.793E+01	n/a	n/a	n/a	n/a
125	1.510E+01	2.287E+01	4.517E+01	6.126E+01	n/a	n/a
150	1.078E+01	1.386E+01	2.077E+01	3.560E+01	n/a	n/a
175	8.553E+00	9.653E+00	1.406E+01	2.230E+01	4.265E+01	n/a
200	7.201E+00	7.354E+00	8.779E+00	1.634E+01	2.343E+01	3.664E+01
225	6.411E+00	6.352E+00	7.023E+00	1.216E+01	1.557E+01	3.024E+01
250	5.916E+00	5.723E+00	6.323E+00	9.850E+00	1.116E+01	1.429E+01
275	5.585E+00	5.465E+00	6.053E+00	8.343E+00	9.647E+00	9.585E+00
300	5.395E+00	5.245E+00	5.740E+00	7.333E+00	8.034E+00	7.164E+00

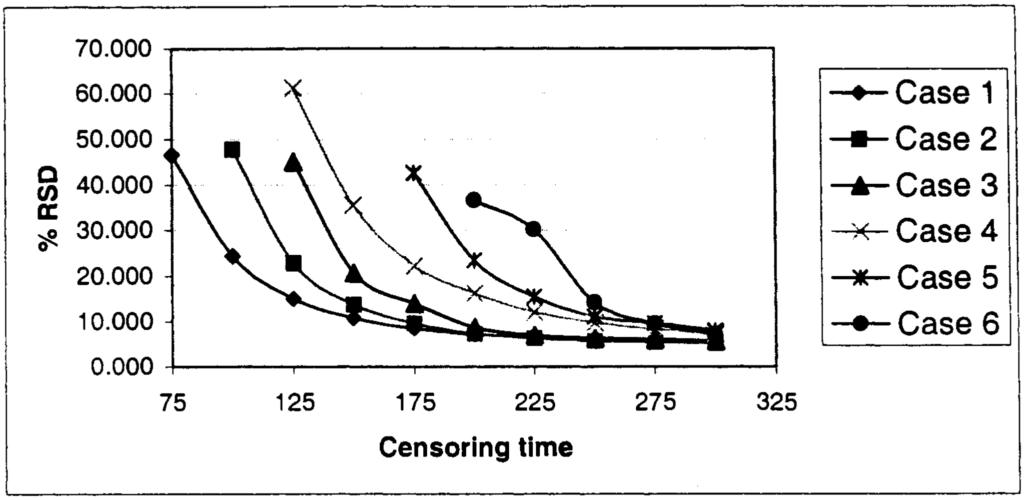


Figure 16. Comparisons of % RSD in estimating $\hat{\alpha}_3$ with different censoring times for scenario 2

Table 15. Comparing % RSD in estimating $\hat{\beta}$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	8.181E+01	n/a	n/a	n/a	n/a	n/a
100	3.528E+01	1.038E+02	n/a	n/a	n/a	n/a
125	2.075E+01	2.951E+01	8.437E+01	1.126E+02	n/a	n/a
150	1.348E+01	1.743E+01	2.323E+01	5.431E+01	n/a	n/a
175	1.020E+01	1.086E+01	1.595E+01	3.057E+01	1.177E+02	n/a
200	1.066E+01	7.904E+00	9.544E+00	2.085E+01	3.257E+01	4.359E+01
225	6.894E+00	6.666E+00	7.490E+00	1.467E+01	1.966E+01	3.434E+01
250	6.314E+00	5.911E+00	6.653E+00	1.139E+01	1.287E+01	1.561E+01
275	5.880E+00	5.592E+00	6.436E+00	9.324E+00	1.052E+01	1.013E+01
300	5.644E+00	5.345E+00	6.020E+00	8.374E+00	8.898E+00	7.385E+00

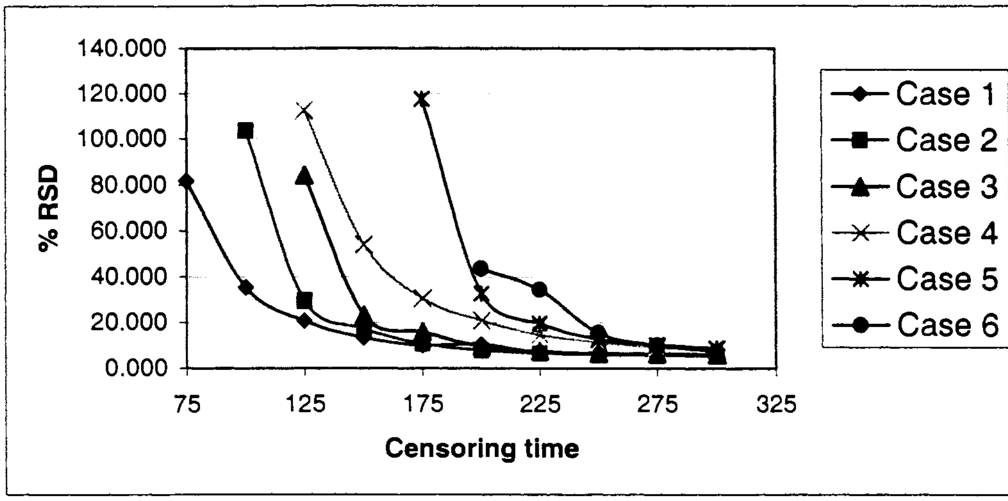


Figure 17. Comparisons of % RSD in estimating $\hat{\beta}$ with different censoring times for scenario 2

Table 16. Comparing % RSD in estimating $\hat{\mu}$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	2.575E+01	n/a	n/a	n/a	n/a	n/a
100	1.073E+01	3.261E+01	n/a	n/a	n/a	n/a
125	5.520E+00	6.592E+00	1.670E+01	3.202E+01	n/a	n/a
150	3.124E+00	2.947E+00	3.721E+00	1.366E+01	n/a	n/a
175	2.118E+00	1.659E+00	1.730E+00	7.001E+00	2.347E+01	n/a
200	1.643E+00	1.130E+00	8.959E-01	4.352E+00	6.242E+00	7.317E+00
225	1.432E+00	1.014E+00	7.247E-01	2.699E+00	3.145E+00	4.848E+00
250	1.362E+00	9.559E-01	6.863E-01	1.995E+00	1.767E+00	1.724E+00
275	1.311E+00	9.351E-01	6.865E-01	1.553E+00	1.242E+00	9.161E-01
300	1.296E+00	9.255E-01	6.649E-01	1.335E+00	9.650E-01	6.411E-01

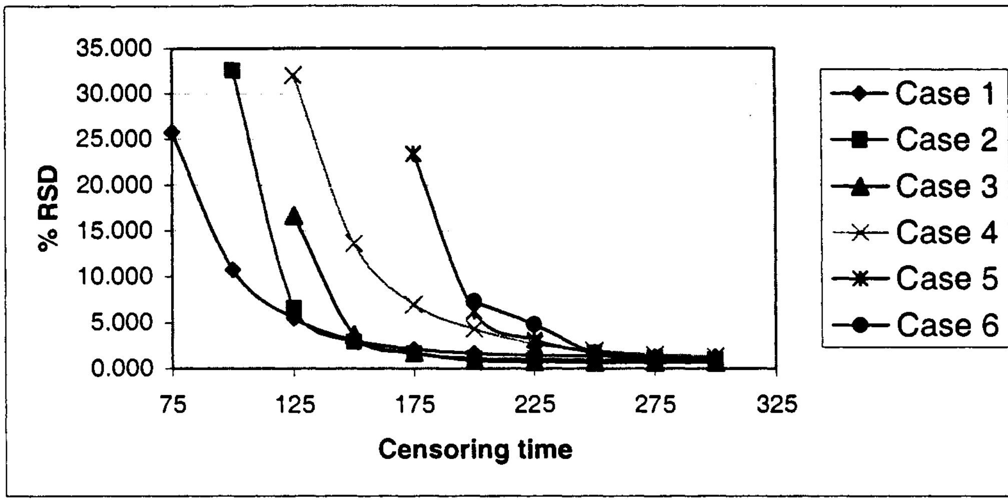


Figure 18. Comparisons of % RSD in estimating $\hat{\mu}$ with different censoring times for scenario 2

Table 17. Comparing % RSD in estimating $\hat{R}$ for scenario 2

Censoring time	Case1	Case2	Case3	Case4	Case5	Case6
75	1.753E+01	n/a	n/a	n/a	n/a	n/a
100	4.356E+00	2.864E+01	n/a	n/a	n/a	n/a
125	2.439E+00	7.953E-01	1.327E+01	4.374E+01	n/a	n/a
150	1.174E+00	6.021E-01	9.028E-02	1.773E+01	n/a	n/a
175	7.814E-01	1.578E-01	1.754E-02	9.777E+00	4.420E+01	n/a
200	5.126E-01	7.637E-02	8.629E-03	6.642E+00	1.237E+01	8.678E+00
225	4.078E-01	7.483E-02	4.987E-03	4.283E+00	5.818E+00	6.047E+00
250	3.607E-01	6.478E-02	4.696E-03	3.285E+00	3.259E+00	2.713E+00
275	3.249E-01	6.049E-02	3.671E-03	2.537E+00	2.380E+00	1.507E+00
300	3.164E-01	5.697E-02	2.105E-03	2.122E+00	1.810E+00	9.851E-01

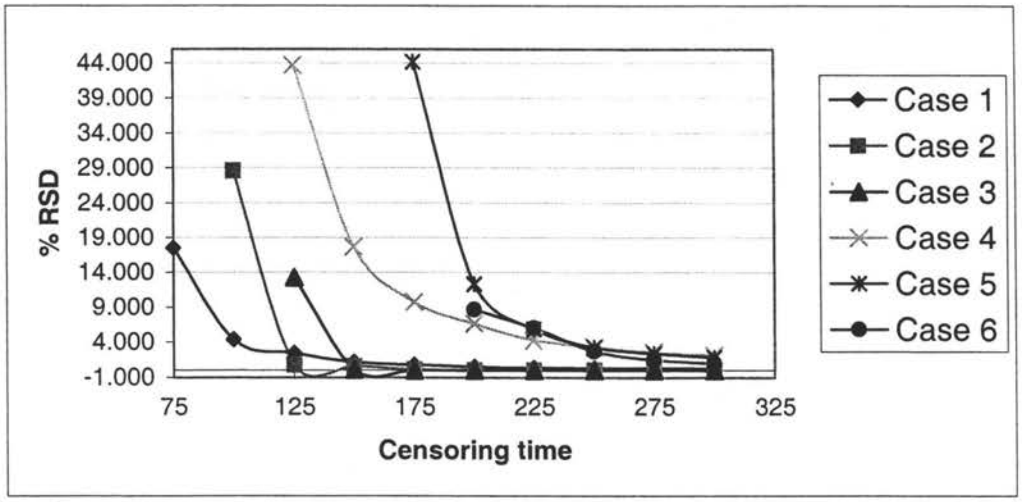


Figure 19. Comparisons of % RSD in estimating $\hat{R}$ with different censoring times for scenario 2

Comparing scenarios

Tables 14-17 and Figures 16-19 illustrate the percent errors and RSDs for $\hat{\mu}$ and $\hat{R}$ at different censoring times for scenario 1 and 2. The reason that we focus on the results for $\hat{\mu}$ and $\hat{R}$ is because these estimates provide a lot more insightful information about the returns than just considering only $\hat{\alpha}_3$ and $\hat{\beta}$. Comparing scenarios 1 and 2, the results suggest that knowledge of the scale parameter contributes greatly to the precision. For the cases that have small expected length of time to return, both errors and RSD gradually decrease and stay constant earlier than those with a higher mean time to return because more observations are taken into account in estimation. The variability of sale distributions affects the width of confidence intervals; the higher the C.V., the wider the confidence interval for large censoring times.

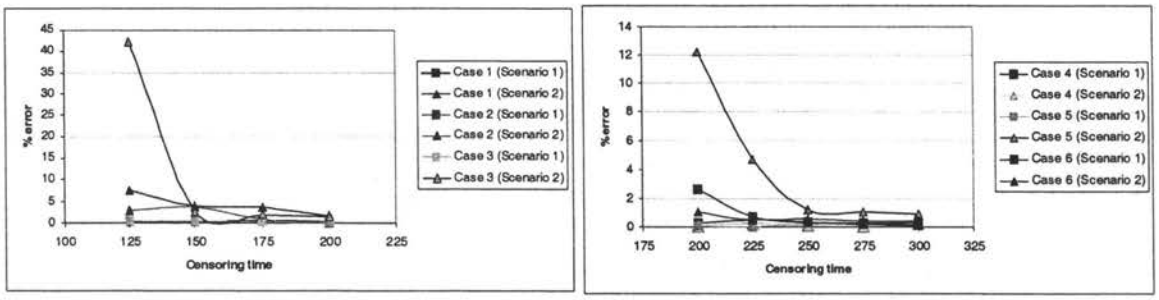


Figure 20. Comparisons of % error in estimating $\hat{\mu}$ with different censoring times for scenarios 1 and 2

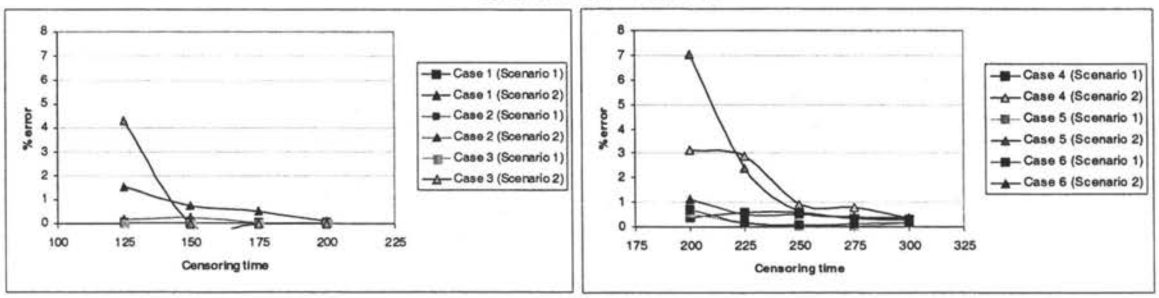


Figure 21. Comparisons of % error in estimating $\hat{R}$ with different censoring times for scenarios 1 and 2

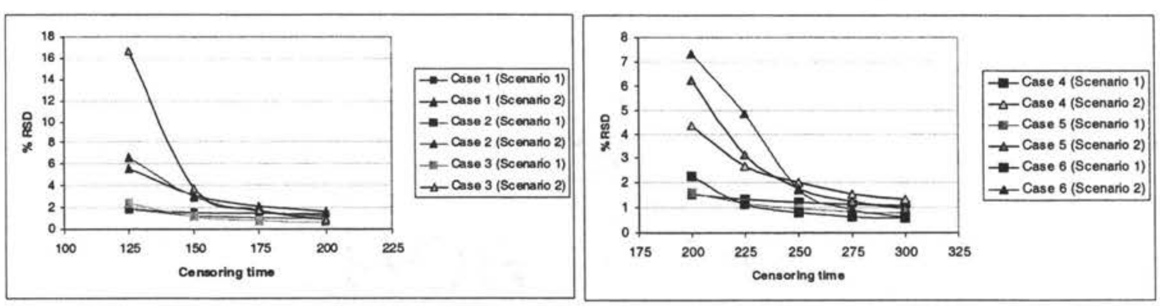


Figure 22. Comparisons of % RSD in estimating $\hat{\mu}$ with different censoring times for scenarios 1 and 2

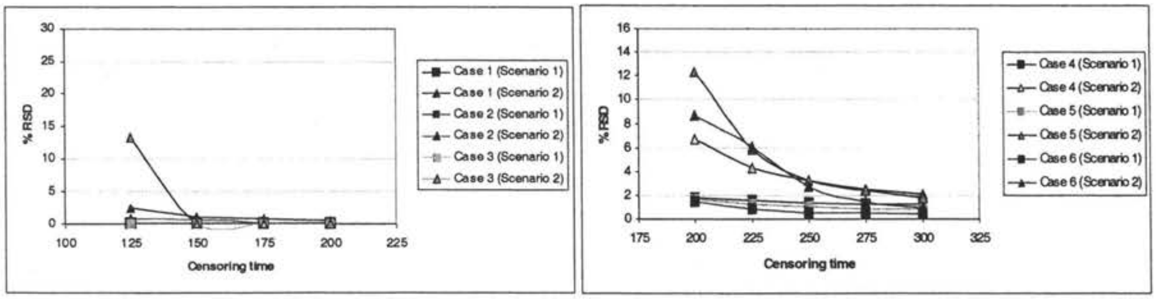


Figure 23. Comparisons of % RSD in estimating $\hat{R}$ with different censoring times for scenarios 1 and 2

CHAPTER 5. SUMMARY AND CONCLUSION

5.1 Summary

The objective of this research is to study the effect of characteristics of time to return distributions on the predictability of the future returns of electronic products. We assumed that information related to the times at which individual units were sold and returned was obtainable and consisted of observations from two independent gamma distributions. Maximum Likelihood Estimation was used to give the point estimates for a multiple censored data set including confidence intervals. We considered two scenarios depending on available knowledge about the scale parameter and furthermore each scenario consisted of six different test cases. The observation data were simulated according to the average useful life of PC (Grenchus, 2002).

5.2 Conclusion

The results suggest that for every case the accuracy and precision of estimates improves as censoring times increase, until reaching the certain time at which precision and accuracy remain approximately constant. The variability of sale time and the mean of the time to return distributions affect the precision in estimation: the higher the variability, the wider the confidence interval of estimates. With the knowledge about return time distributions, the confidence interval width is smaller than without prior knowledge. Nonetheless, without prior knowledge, we can still use estimates of mean return time and proportion returned before obsolescence that have the percent error considerably smaller compared to errors in estimates of shape and scale parameters.

5.3 Future research

An interesting extension to be considered is to estimate distributions of Y_2, the time that an item is in use, rather than $T = Y_1 + Y_2$, the return time, because the return time distribution could be derived from returns the stream but the time from sale to return of an item is actually difficult to obtain and it would be interesting to know how long an items spends in use. It also would be interesting to apply other estimation approaches to evaluate performance of predictability with different amounts of variability in the data by measuring the average deviation from the true values comparing with MLE approach. One example for a estimation approach to explore is Bayesian analysis. The Bayesian approach takes the benefit of historical data sets that could lead to the improvement in estimation and it also relaxes the assumption of asymptotic normality that might not hold with a highly censored data set. Nonetheless, we need extremely intensive computation effort to perform the Bayesian updates (Punt and Hilborn, 2003). Another possible extension is to consider the case that not all units will be returned. The assumption in this thesis that all units will be returned might not exactly fit in a remanufacturing environment where only a portion of units sold will be returned.

APPENDIX A. SCENARIO 1 DATA

Scenario 1. Assuming prior kn wledge of β is available

Table A1. The estimated parar eters and variances for case 1 (scenario 1)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\mu})$
75	8.95	741.05	0.981	8.038	200.959	2.324E-05	9.743E-02	6.089E+01
100	37.86	712.14	0.981	8.033	200.813	7.921E-06	3.591E-02	2.244E+01
125	99.72	650.28	0.982	8.019	200.466	4.378E-06	2.056E-02	1.285E+01
150	190.48	559.52	0.982	8.024	200.599	3.253E-06	1.509E-02	9.433E+00
175	300.19	449.81	0.981	8.031	200.771	2.718E-06	1.263E-02	7.893E+00
200	410.23	339.77	0.982	8.025	200.614	2.418E-06	1.137E-02	7.108E+00
225	506.09	243.91	0.982	8.018	200.445	2.254E-06	1.072E-02	6.701E+00
250	582.73	167.27	0.982	8.017	200.430	2:178E-06	1.039E-02	6.495E+00
275	641.77	108.23	0.982	8.019	200.484	2.147E-06	1.022E-02	6.389E+00
300	682.86	67.14	0.982	8.017	200.433	2.122E-06	1.013E-02	6.330E+00
325	709.35	40.65	0.982	8.017	200.428	2.112E-06	1.008E-02	6.302E+00
350	725.98	24.02	0.982	8.017	200.431	2.108E-06	1.006E-02	6.287E+00
375	736.21	13.79	**0.982**	8.017	200.419		1.005E-02	6.280E+00
400	742.37	7.63	**0.982**	8.017	200.417		1.004E-02	6.276E+00
425	745.88	4.12	**0.982**	8.017	200.413		1.004E-02	6.274E+00
450	747.79	2.21	**0.982**	8.016	200.404		1.004E-02	6.273E+00
475	748.77	1.23	**0.982**	8.016	200.404		1.004E-02	6.273E+00
500	749.31	0.69	**0.982**	8.016	200.408		1.004E-02	6.273E+00
700	750.00	0.00	**0.982**	8.016	200.409		1.004E-02	6.273E+00

Table A2. 90% confidence intervals for es*imated parameters from case 1 (scenario 1)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
75	0.973	0.989	7.525	8.552	188.123	213.795
100	0.977	0.986	7.721	8.344	193.020	208.606
125	0.978	0.985	7.783	8.255	194.569	206.364
150	0.979	0.985	7.822	8.226	195.546	205.651
175	0.979	0.984	7.846	8.216	196.149	205.393
200	0.979	0.984	7.849	8.200	196.228	205.000
225	0.979	0.984	7.847	8.188	196.186	204.703
250	0.979	0.984	7.850	8.185	196.238	204.622
275	0.979	0.984	7.853	8.186	196.326	204.642
300	0.979	0.984	7.852	8.183	196.294	204.572
325	0.979	0.984	7.852	8.182	196.299	204.557
350	0.979	0.984	7.852	8.182	196.306	204.556
375			7.965	8.069	196.297	204.541
400			7.852	8.182	196.296	204.538
425			7.852	8.181	196.292	204.533
450			7.851	8.181	196.284	204.524
475			7.851	8.181	196.284	204.524
500			7.852	8.181	196.288	204.528
700			7.852	8.181	196.289	204.529

Table A3. The estimated parameters and variances for case 2 (scenario 1)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\mu})$
100	5.98	744.02	0.998	16.255	203.185	1.250E-06	3.365E-01	5.258E+01
125	35.42	714.58	0.998	16.056	200.698	1.780E-07	8.159E-02	1.275E+01
150	117.30	632.70	0.998	16.024	200.305	8.170E-08	4.081E-02	6.377E+00
175	246.52	503.48	0.998	16.035	200.439	5.676E-08	2.848E-02	4.450E+00
200	397.78	352.22	0.998	16.017	200.218	4.930E-08	2.374E-02	3.710E+00
225	532.33	217.67	0.998	16.009	200.106	4.490E-08	2.194E-02	3.428E+00
250	631.50	118.50	0.998	16.010	200.124	4.290E-08	2.117E-02	3.307E+00
275	691.46	58.54	0.998	16.015	200.184	4.130E-08	2.085E-02	3.259E+00
300	723.72	26.28	0.998	16.010	200.124	4.160E-08	2.074E-02	3.250E+00
325	739.16	10.84	0.998	16.009	200.114	4.140E-08	2.070E-02	3.234E+00
350	745.91	4.09	0.998	16.008	200.104	4.140E-08	2.069E-02	3.232E+00
375	748.33	1.67	**0.998**	16.008	200.099		2.068E-02	3.231E+00
400	749.45	0.55	**0.998**	16.007	200.091		2.068E-02	3.231E+00
425	749.87	0.13	**0.998**	16.008	200.101		2.068E-02	3.231E+00
450	749.96	0.04	**0.998**	16.009	200.106		2.068E-02	3.231E+00
475	750.00	0.00	**0.998**	16.009	200.106		2.068E-02	3.231E+00

Table A4. 90% confidence intervals for estimated parameters from case 2 (scenario 1)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
100	0.996	0.999	15.301	17.209	191.257	215.113
125	0.997	0.999	15.586	16.526	194.824	206.571
150	0.998	0.999	15.692	16.357	196.151	204.459
175	0.998	0.998	15.757	16.313	196.969	203.909
200	0.998	0.998	15.764	16.271	197.049	203.386
225	0.998	0.998	15.765	16.252	197.060	203.152
250	0.998	0.998	15.771	16.249	197.132	203.115
275	0.998	0.998	15.777	16.252	197.214	203.153
300	0.998	0.998	15.773	16.247	197.158	203.089
325	0.998	0.998	15.772	16.246	197.155	203.072
350	0.998	0.998	15.772	16.245	197.146	203.061
375			15.771	16.244	197.142	203.056
400			15.771	16.244	197.134	203.048
425			15.772	16.245	197.144	203.058
450			15.772	16.245	197.149	203.063
475			15.772	16.245	197.149	203.063

Table A5. The estimated parameters and variances for case 3 (scenario 1)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\mu})$
125	7.10	742.90	0.999	31.828	198.923	5.107E-10	5.922E-01	2.313E+01
150	50.66	699.34	0.999	31.837	198.979	4.349E-11	1.312E-01	5.126E+00
175	187.01	562.99	0.999	31.885	199.278	2.234E-11	6.586E-02	2.573E+00
200	392.55	357.45	0.999	31.892	199.324	1.640E-11	4.880E-02	1.906E+00
225	577.89	172.11	0.999	31.902	199.388	1.473E-11	4.377E-02	1.710E+00
250	686.55	63.45	0.999	31.911	199.445	1.440E-11	4.234E-02	1.654E+00
275	731.80	18.20	0.999	31.914	199.463	1.429E-11	4.198E-02	1.640E+00
300	745.75	4.25	0.999	31.912	199.449	1.423E-11	4.190E-02	1.637E+00
325	749.35	0.65	0.999	31.912	199.450	1.423E-11	4.189E-02	1.636E+00
350	749.91	0.09	0.999	31.912	199.450	1.422E-11	4.189E-02	1.636E+00
375	749.99	0.01	1.000	31.912	199.448		4.189E-02	1.636E+00
400	750.00	0.00	1.000	31.912	199.449		4.189E-02	1.636E+00

Table A6. 90% confidence intervals for estimated parameters from case 3 (scenario 1)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
125	0.999	1.000	30.562	33.094	191.011	206.835
150	0.999	0.999	31.241	32.433	195.254	202.703
175	0.999	0.999	31.751	32.018	196.640	201.917
200	0.999	0.999	31.777	32.007	197.053	201.596
225	0.999	0.999	31.558	32.246	197.237	201.539
250	0.999	0.999	31.573	32.250	197.329	201.561
275	0.999	0.999	31.577	32.251	197.357	201.570
300	0.999	0.999	31.575	32.249	197.345	201.554
325	0.999	0.999	31.575	32.249	197.346	201.554
350	0.999	0.999	31.575	32.249	197.345	201.554
375			31.575	32.248	197.344	201.552
400			31.575	32.248	197.345	201.553

Table A7. The estimated parameters and variances for case 4 (scenario 1)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\mu})$
125	4.15	745.85	0.809	12.064	301.605	1.798E-03	2.932E-01	1.832E+02
150	14.96	735.04	0.809	12.063	301.575	6.580E-04	1.097E-01	6.854E+01
175	40.54	709.46	0.808	12.083	302.068	3.384E-04	5.703E-02	3.564E+01
200	85.09	664.91	0.812	12.036	300.910	2.049E-04	3.569E-02	2.230E+01
225	148.58	601.42	0.810	12.060	301.508	1.556E-04	2.682E-02	1.676E+01
250	227.81	522.19	0.810	12.032	300.795	1.275E-04	2.200E-02	1.375E+01
275	315.73	434.27	0.813	12.033	300.825	1.096E-04	1.916E-02	1.198E+01
300	403.80	346.20	0.813	12.031	300.778	1.007E-04	1.762E-02	1.101E+01
325	485.42	264.58	0.814	12.020	300.493	9.471E-05	1.667E-02	1.042E+01
350	554.75	195.25	0.813	12.023	300.563	9.181E-05	1.614E-02	1.009E+01
375	611.24	138.76	**0.815**	12.024	300.593		1.582E-02	9.888E+00
400	654.59	95.41	**0.815**	12.018	300.458		1.562E-02	9.760E+00
425	686.32	63.68	**0.815**	12.021	300.528		1.551E-02	9.696E+00
450	708.26	41.74	**0.815**	12.020	300.490		1.545E-02	9.654E+00
475	724.07	25.93	**0.815**	12.029	300.713		1.541E-02	9.633E+00
500	733.85	16.15	**0.815**	12.020	300.498		1.539E-02	9.620E+00
525	740.23	9.77	**0.815**	12.020	300.506		1.538E-02	9.614E+00
550	744.28	5.72	**0.815**	12.019	300.485		1.538E-02	9.609E+00
575	746.75	3.25	**0.815**	12.019	300.486		1.537E-02	9.607E+00
600	748.25	1.75	**0.815**	12.019	300.470		1.537E-02	9.606E+00
700	750.00	0.00	**0.815**	12.019	300.485		1.537E-02	9.606E+00

Table A8. 90% confidence intervals for estimated parameters from case 4 (scenario 1)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
125	0.739	0.879	11.174	12.955	279.338	323.872
150	0.767	0.852	11.518	12.608	287.956	315.194
175	0.778	0.839	11.690	12.476	292.246	311.889
200	0.738	0.887	11.726	12.347	293.141	308.679
225	0.790	0.831	11.791	12.330	294.772	308.243
250	0.792	0.829	11.788	12.276	294.696	306.894
275	0.795	0.830	11.805	12.261	295.132	306.518
300	0.796	0.829	11.813	12.249	295.318	306.237
325	0.798	0.830	11.811	12.229	295.183	305.802
350	0.798	0.829	11.816	12.229	295.338	305.787
375			11.818	12.229	295.420	305.765
400			11.813	12.223	295.318	305.597
425			11.817	12.226	295.405	305.650
450			11.815	12.224	295.379	305.601
475			11.824	12.233	295.607	305.818
500			11.816	12.224	295.395	305.600
525			11.816	12.224	295.405	305.606
550			11.815	12.223	295.386	305.584
575			11.816	12.223	295.387	305.585
600			11.815	12.223	295.372	305.568
700			12.019	12.019	295.387	305.583

Table A9. The estimated parameters and variances for case 5 (scenario 1)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\mu})$
175	7.14	742.86	0.882	24.194	302.420	1.097E-03	4.378E-01	6.840E+01
200	27.35	722.65	0.881	23.983	299.785	2.095E-04	1.468E-01	2.294E+01
225	76.18	673.82	0.887	23.999	299.991	1.227E-04	7.930E-02	1.239E+01
250	160.86	589.14	0.886	24.051	300.639	7.846E-05	5.356E-02	8.369E+00
275	272.65	477.35	0.886	24.033	300.410	6.180E-05	4.219E-02	6.593E+00
300	396.62	353.38	0.884	24.009	300.109	5.515E-05	3.651E-02	5.705E+00
325	510.59	239.41	0.884	24.051	300.639	5.117E-05	3.391E-02	5.298E+00
350	599.77	150.23	0.885	24.061	300.766	4.304E-05	3.256E-02	5.088E+00
375	663.63	86.37	**0.885**	24.064	300.794		3.193E-02	4.989E+00
400	704.61	45.39	**0.885**	24.050	300.629		3.163E-02	4.943E+00
425	727.04	22.96	**0.885**	24.056	300.698		3.150E-02	4.922E+00
450	739.08	10.92	**0.885**	24.058	300.729		3.145E-02	4.914E+00
475	745.06	4.94	**0.885**	24.058	300.721		3.173E-02	4.911E+00
500	747.91	2.09	**0.885**	24.057	300.710		3.142E-02	4.909E+00
700	750.00	0.00	**0.885**	24.056	300.703		3.142E-02	4.909E+00

Table A10. 90% confidence intervals for estimated parameters from case 5 (scenario 1)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
175	0.828	0.937	23.105	25.282	288.815	316.025
200	0.857	0.904	23.352	24.613	291.906	307.664
225	0.868	0.905	23.536	24.463	294.201	305.782
250	0.871	0.900	23.670	24.432	295.880	305.398
275	0.873	0.899	23.695	24.371	296.186	304.634
300	0.872	0.896	23.694	24.323	296.180	304.038
325	0.872	0.896	23.748	24.354	296.852	304.425
350	0.874	0.896	23.764	24.358	297.056	304.477
375			23.770	24.357	297.120	304.468
400			23.758	24.343	296.972	304.286
425			23.764	24.348	297.048	304.347
450			23.767	24.350	297.082	304.375
475			23.765	24.351	297.076	304.367
500			23.765	24.348	297.065	304.355
700			23.765	24.348	297.058	304.347

Table A11. The estimated parameters and variances for case 6 (scenario 1)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\mu})$
200	3.62	746.38	0.957	49.265	307.907	1.887E-04	1.238E+00	4.834E+01
225	24.08	725.92	0.952	48.343	302.146	6.469E-05	2.894E-01	1.130E+01
250	89.86	660.14	0.951	48.160	301.000	2.894E-05	1.397E-01	5.459E+00
275	220.75	529.25	0.949	48.106	300.661	2.010E-05	9.164E-02	3.580E+00
300	391.63	358.37	0.949	48.059	300.371	1.595E-05	7.403E-02	2.892E+00
325	548.25	201.75	0.948	48.032	300.200	1.456E-05	6.679E-02	2.609E+00
350	655.91	94.09	0.948	48.022	300.139	1.397E-05	6.411E-02	2.504E+00
375	712.76	37.24	**0.950**	48.039	300.244		6.334E-02	2.474E+00
400	737.96	12.04	**0.950**	48.040	300.252		6.308E-02	2.464E+00
425	746.64	3.36	**0.950**	48.040	300.248		6.301E-02	2.461E+00
450	749.23	0.77	**0.950**	48.037	300.233		6.300E-03	2.461E+00
475	749.81	0.19	**0.950**	48.039	300.242		6.300E-02	2.461E+00
500	749.94	0.06	**0.950**	48.039	300.242		6.300E-02	2.461E+00
700	750.00	0.00	**0.950**	48.039	300.242		6.300E-02	2.461E+00

Table A12. 90% confidence intervals for estimated parameters from case 6 (scenario 1)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
200	0.935	0.980	47.435	51.095	296.470	319.344
225	0.939	0.966	47.458	49.228	296.616	307.676
250	0.943	0.961	47.545	48.775	297.157	304.843
275	0.942	0.957	47.608	48.604	297.548	303.773
300	0.943	0.956	47.612	48.507	297.574	303.169
325	0.942	0.955	47.607	48.457	297.543	302.857
350	0.942	0.955	47.606	48.439	297.536	302.742
375			47.625	48.453	297.657	302.832
400			47.627	48.453	297.670	302.834
425			47.627	48.453	297.667	302.828
450			47.907	48.168	297.652	302.813
475			47.626	48.452	297.661	302.822
500			47.626	48.452	297.661	302.822
700			47.626	48.452	297.661	302.822

APPENDIX B. SCENARIO 2 DATA

Scenario 2. Assuming without prior knowledge of β

Table B1. The estimated parameters for case 1 (scenario 2)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\beta}$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\beta})$	$V(\hat{\mu})$
75	9.53	740.47	0.935	8.640	30.387	262.534	2.689E-02	1.624E+01	6.180E+02	4.571E+03
100	37.96	712.04	0.967	7.844	27.845	218.413	1.775E-03	3.648E+00	9.650E+01	5.494E+02
125	101.63	648.37	0.967	7.630	28.205	215.197	5.562E-04	1.327E+00	3.424E+01	1.411E+02
150	192.70	557.30	0.975	7.740	26.832	207.678	1.310E-04	6.968E-01	1.308E+01	4.210E+01
175	302.10	447.90	0.977	7.917	26.207	207.471	5.826E-05	4.584E-01	7.143E+00	1.931E+01
200	411.26	338.74	0.981	8.159	24.929	203.384	2.528E-05	3.452E-01	7.061E+00	1.116E+01
225	507.28	242.72	0.982	8.197	24.942	204.445	1.604E-05	2.762E-01	2.957E+00	8.572E+00
250	585.63	164.37	0.982	8.201	24.595	201.694	1.255E-05	2.354E-01	2.412E+00	7.548E+00
275	642.75	107.25	0.983	8.272	24.344	201.365	1.020E-05	2.134E-01	2.049E+00	6.974E+00
300	682.94	67.06	0.982	8.221	24.511	201.508	9.660E-06	1.967E-01	1.914E+00	6.815E+00
325	709.04	40.96	0.982	8.221	24.512	201.521	9.071E-06	1.875E-01	1.801E+00	6.695E+00
350	725.02	24.98	0.983	8.259	24.366	201.234	8.432E-06	1.828E-01	1.705E+00	6.579E+00
375	735.90	14.10	**0.981**	8.272	24.325	201.224		1.800E-01	1.662E+00	6.535E+00
400	742.11	7.89	**0.981**	8.274	24.319	201.219		1.783E-01	1.641E+00	6.518E+00
425	745.74	4.26	**0.981**	8.261	24.358	201.226		1.768E-01	1.636E+00	6.522E+00
450	747.72	2.28	**0.981**	8.253	24.385	201.237		1.759E-01	1.634E+00	6.527E+00
475	748.70	1.30	**0.981**	8.257	24.367	201.208		1.757E-01	1.626E+00	6.519E+00
500	749.36	0.64	**0.981**	8.264	24.346	201.187		1.754E-01	1.618E+00	6.510E+00
700	750.00	0.00	**0.981**	8.255	24.371	201.173		1.753E-01	1.624E+00	6.517E+00

Table B2. 90% confidence intervals for estimated parameters from case1 (scenario 2)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\beta}$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
75	0.666	1.000	2.011	15.269	0.000	71.281	151.313	373.756
100	0.898	1.000	4.702	10.986	11.686	44.004	179.856	256.969
125	0.928	1.000	5.735	9.525	18.579	37.831	195.656	234.738
150	0.956	0.994	6.367	9.113	20.883	32.781	197.004	218.352
175	0.964	0.989	6.803	9.030	21.811	30.604	200.243	214.699
200	0.973	0.989	7.192	9.125	20.558	29.300	197.887	208.880
225	0.975	0.989	7.332	9.061	22.113	27.770	199.629	209.261
250	0.976	0.988	7.402	8.999	22.041	27.150	197.174	206.213
275	0.977	0.988	7.512	9.032	21.989	26.699	197.021	205.710
300	0.977	0.987	7.492	8.951	22.235	26.786	197.213	205.802
325	0.977	0.987	7.509	8.934	22.304	26.720	197.265	205.778
350	0.978	0.987	7.555	8.962	22.218	26.514	197.014	205.453
375			7.574	8.970	22.205	26.446	197.019	205.429
400			7.580	8.969	22.212	26.426	197.020	205.419
425			7.570	8.953	22.253	26.462	197.025	205.427
450			7.563	8.943	22.282	26.487	197.034	205.439
475			7.568	8.947	22.269	26.465	197.008	205.408
500			7.575	8.953	22.253	26.438	196.990	205.385
700			7.566	8.943	22.275	26.467	196.974	205.373

Table B3. The estimated parameters for case 2 (scenario 2)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\beta}$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\beta})$	$V(\hat{\mu})$
100	6.31	743.69	0.929	11.503	22.610	260.083	7.074E-02	3.039E+01	5.506E+02	7.195E+03
125	37.10	712.90	0.996	16.000	12.870	205.921	6.279E-05	1.338E+01	1.442E+01	1.842E+02
150	116.94	633.06	0.996	16.206	12.800	207.444	3.594E-05	5.049E+00	4.980E+00	3.738E+01
175	248.80	501.20	0.997	15.468	13.036	201.642	2.476E-06	2.229E+00	2.003E+00	1.119E+01
200	402.31	347.69	0.998	16.282	12.253	199.510	5.812E-07	1.434E+00	9.380E-01	5.078E+00
225	537.18	212.82	0.998	15.618	12.809	200.052	5.575E-07	9.841E-01	7.291E-01	4.115E+00
250	633.68	116.32	0.998	15.655	12.782	200.098	4.179E-07	8.028E-01	5.709E-01	3.658E+00
275	693.64	56.36	0.998	15.645	12.786	200.036	3.644E-07	7.309E-01	5.112E-01	3.499E+00
300	724.66	25.34	0.998	15.707	12.735	200.025	3.232E-07	6.787E-01	4.634E-01	3.427E+00
325	739.88	10.12	0.998	15.769	12.685	200.035	3.003E-07	6.553E-01	4.381E-01	3.392E+00
350	746.05	3.95	0.998	15.828	12.635	199.997	2.847E-07	6.478E-01	4.252E-01	3.368E+00
375	748.43	1.57	**0.998**	15.922	12.562	200.008		6.425E-01	4.113E-01	3.345E+00
400	749.38	0.62	**0.998**	15.845	12.626	200.058		6.419E-01	4.197E-01	3.362E+00
425	749.82	0.18	**0.998**	15.787	12.673	200.071		6.412E-01	4.255E-01	3.376E+00
450	749.94	0.06	**0.998**	15.649	12.800	200.303		6.406E-01	4.425E-01	3.412E+00
475	750.00	0.00	**0.998**	16.549	12.089	200.056		6.164E-01	4.141E-01	3.224E+00

Table B4. 90% confidence intervals for estimated parameters from case 2 (scenario 2)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\beta}$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
100	0.491	1.366	2.434	20.572	0.000	61.209	120.547	399.619
125	0.983	1.009	9.981	22.018	6.623	19.118	183.593	228.250
150	0.986	1.005	12.510	19.902	9.129	16.471	197.387	217.502
175	0.995	1.000	13.012	17.924	10.708	15.364	196.140	207.144
200	0.997	1.000	14.312	18.252	10.660	13.847	195.803	203.217
225	0.997	0.999	13.986	17.250	11.404	14.214	196.715	203.389
250	0.997	0.999	14.181	17.129	11.539	14.025	196.952	203.244
275	0.997	0.999	15.200	16.090	11.610	13.962	196.959	203.113
300	0.997	0.999	14.351	17.062	11.615	13.855	196.980	203.071
325	0.997	0.999	14.438	17.101	11.596	13.774	197.006	203.065
350	0.997	0.999	14.504	17.152	11.563	13.708	196.978	203.016
375			14.603	17.241	11.507	13.617	196.999	203.016
400			14.527	17.163	11.560	13.692	197.042	203.074
425			14.470	17.104	11.600	13.746	197.048	203.093
450			14.332	16.965	11.706	13.894	197.265	203.342
475			14.332	16.965	11.706	13.894	197.265	203.342

Table B5. The estimated parameters for case 3 (scenario 2)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\beta}$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\beta})$	$V(\hat{\mu})$
125	5.10	744.90	0.957	22.738	12.516	284.587	1.613E-02	1.055E+02	1.115E+02	2.260E+03
150	49.05	700.95	0.999	29.541	6.949	205.265	8.148E-07	3.764E+01	2.605E+00	5.834E+01
175	184.94	565.06	0.999	29.297	6.959	203.887	3.077E-08	1.696E+01	1.233E+00	1.244E+01
200	391.52	358.49	0.999	29.186	6.958	203.091	7.445E-09	6.566E+00	4.411E-01	3.311E+00
225	574.53	175.47	0.999	29.110	6.957	202.534	2.486E-09	4.179E+00	2.716E-01	2.154E+00
250	683.42	66.58	0.999	29.066	6.970	202.592	2.205E-09	3.377E+00	2.151E-01	1.933E+00
275	730.46	19.55	0.999	29.056	6.886	200.071	1.347E-09	3.094E+00	1.964E-01	1.886E+00
300	745.85	4.15	0.999	30.010	6.749	202.528	4.428E-10	2.967E+00	1.650E-01	1.813E+00
325	749.18	0.82	0.999	32.504	6.213	201.964	1.660E-10	2.885E+00	1.129E-01	1.658E+00
350	749.90	0.10	0.999	32.919	6.092	200.556	1.627E-10	2.874E+00	9.997E-02	1.624E+00
375	750.00	0.00	**0.999**	32.937	6.088	200.536	1.627E-10	2.874E+00	9.971E-02	1.622E+00

Table B6. 90% confidence intervals for estimated parameters from case 3 (scenario 2)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\beta}$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
125	0.748	1.000	5.844	39.631	0.000	29.888	206.389	362.785
150	0.998	1.000	19.448	39.634	4.294	9.604	192.701	217.830
175	0.999	1.000	22.522	36.072	5.133	8.786	198.085	209.690
200	0.999	1.000	24.971	33.401	5.866	8.051	200.098	206.084
225	0.999	0.999	25.747	32.473	6.100	7.815	200.120	204.949
250	0.999	0.999	26.043	32.089	6.207	7.733	200.305	204.879
275	0.999	0.999	26.163	31.950	6.157	7.615	197.811	202.330
300	0.999	0.999	27.176	32.843	6.080	7.417	200.313	204.743
325	0.999	0.999	29.710	35.298	5.661	6.766	199.846	204.083
350	0.999	0.999	30.131	35.708	5.572	6.613	198.460	202.652
375			30.149	35.726	5.569	6.608	198.440	202.631

Table B7. The estimated parameters for case 4 (scenario 2)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\beta}$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\beta})$	$V(\hat{\mu})$
125	3.80	746.20	0.850	14.666	21.864	320.646	1.381E-01	8.071E+01	6.063E+02	1.054E+04
150	15.68	734.32	0.852	14.269	22.155	316.133	2.279E-02	2.581E+01	1.448E+02	1.866E+03
175	40.27	709.73	0.858	13.888	21.739	301.903	7.034E-03	9.589E+00	4.416E+01	4.467E+02
200	83.68	666.32	0.841	13.020	23.033	299.884	3.119E-03	4.524E+00	2.306E+01	1.703E+02
225	146.95	603.05	0.839	12.872	23.241	299.173	1.291E-03	2.450E+00	1.162E+01	6.520E+01
250	227.48	522.52	0.823	12.204	24.540	299.480	7.306E-04	1.445E+00	7.818E+00	3.568E+01
275	316.64	433.36	0.822	12.155	24.670	299.855	4.347E-04	1.028E+00	5.291E+00	2.169E+01
300	405.26	344.74	0.818	12.004	25.063	300.851	3.012E-04	7.748E-01	4.404E+00	1.613E+01
325	484.67	265.33	0.817	11.966	25.169	301.170	2.299E-04	6.332E-01	3.238E+00	1.331E+01
350	553.45	196.55	0.816	11.921	25.330	301.968	1.910E-04	5.414E-01	2.756E+00	1.195E+01
375	611.32	138.68	**0.815**	11.889	25.378	301.715		4.814E-01	2.413E+00	1.117E+01
400	653.41	96.59	**0.815**	11.975	25.143	301.086		4.462E-01	2.125E+00	1.060E+01
425	685.93	64.07	**0.815**	12.084	24.862	300.446		4.271E-01	1.922E+00	1.020E+01
450	708.71	41.29	**0.815**	12.093	24.837	300.364		4.101E-01	1.827E+00	1.007E+01
475	723.57	26.43	**0.815**	12.080	24.878	300.521		3.996E-01	1.780E+00	1.002E+01
500	734.11	15.89	**0.815**	12.097	24.829	300.364		3.923E-01	1.730E+00	9.948E+00
525	742.12	7.88	**0.815**	12.124	24.763	300.224		3.875E-01	1.691E+00	9.892E+00
550	744.65	5.35	**0.815**	12.126	24.746	300.076		3.853E-01	1.680E+00	9.885E+00
575	746.94	3.06	**0.815**	12.125	24.763	300.262		3.831E-01	1.670E+00	9.876E+00
600	748.37	1.63	**0.815**	12.141	24.735	300.301		3.822E-01	1.658E+00	9.860E+00
700	750.00	0.00	**0.815**	12.100	24.808	300.179		3.817E-01	1.673E+00	9.888E+00

Table B8. 90% confidence intervals for estimated parameters from case 4 (scenario 2)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\beta}$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
125	0.238	1.000	0.000	29.444	0.000	62.367	151.727	489.566
150	0.603	1.000	5.913	22.626	2.363	41.947	245.074	387.191
175	0.720	0.996	8.794	18.982	10.808	32.670	267.136	336.670
200	0.749	0.933	9.521	16.519	15.134	30.932	278.415	321.352
225	0.780	0.898	10.298	15.447	17.634	28.849	285.890	312.456
250	0.778	0.867	10.226	14.181	19.940	29.139	289.654	309.306
275	0.788	0.856	10.487	13.823	20.886	28.453	292.193	307.517
300	0.789	0.847	10.556	13.452	21.610	28.515	294.245	307.458
325	0.792	0.842	10.657	13.275	22.209	28.129	295.168	307.171
350	0.793	0.839	10.711	13.132	22.599	28.061	296.282	307.653
375			10.748	13.030	22.822	27.933	296.217	307.214
400			10.876	13.074	22.745	27.541	295.731	306.440
425			11.009	13.159	22.581	27.143	295.191	305.701
450			11.040	13.147	22.614	27.061	295.144	305.583
475			11.040	13.119	22.683	27.073	295.314	305.729
500			11.067	13.128	22.665	26.992	295.175	305.552
525			11.100	13.148	22.624	26.902	295.050	305.398
550			11.105	13.147	22.614	26.878	294.904	305.248
575			11.107	13.144	22.637	26.888	295.092	305.431
600			11.124	13.158	22.617	26.853	295.136	305.467
700			11.084	13.116	22.681	26.936	295.007	305.352

Table B9. The estimated parameters for case 5 (scenario 2)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\beta}$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\beta})$	$V(\hat{\mu})$
175	6.93	743.07	0.758	21.013	17.27860	363.075	1.124E-01	8.030E+01	4.138E+02	7.260E+03
200	28.13	721.87	0.823	21.948	15.34420	336.770	1.037E-02	2.644E+01	2.497E+01	4.419E+02
225	76.53	673.47	0.864	23.134	13.57650	314.076	2.528E-03	1.298E+01	7.128E+00	9.755E+01
250	161.74	588.26	0.880	23.554	12.88620	303.520	8.221E-04	6.915E+00	2.749E+00	2.876E+01
275	272.54	477.46	0.882	23.954	12.65460	303.128	4.400E-04	5.340E+00	1.773E+00	1.418E+01
300	396.88	353.12	0.883	23.964	12.63440	302.772	2.551E-04	3.706E+00	1.264E+00	8.537E+00
325	510.30	239.70	0.884	23.931	12.63410	302.345	1.545E-04	2.577E+00	7.991E-01	6.273E+00
350	600.77	149.23	0.885	23.923	12.63360	302.234	1.210E-04	2.133E+00	6.519E-01	5.559E+00
375	665.23	84.77	**0.887**	23.911	12.63095	302.017		1.898E+00	5.711E-01	5.268E+00
400	704.62	45.38	**0.887**	23.923	12.62994	302.147		1.730E+00	5.155E-01	5.158E+00
425	727.16	22.84	**0.887**	23.926	12.63027	302.206		1.643E+00	4.877E-01	5.104E+00
450	739.61	10.39	**0.887**	23.925	12.62656	302.174		1.604E+00	4.754E-01	5.079E+00
475	745.64	4.36	**0.887**	23.925	12.62775	302.120		1.582E+00	4.691E-01	5.070E+00
500	748.38	1.63	**0.887**	24.052	12.57210	302.382		1.569E+00	4.565E-01	5.039E+00
700	750.00	0.00	**0.887**	24.287	12.37980	300.662		1.553E+00	4.121E-01	4.950E+00

Table B10. 90% confidence intervals for estimated parameters from case 5 (scenario 2)

S	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\beta}$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
175	0.207	1.000	6.272	35.754	0.000	50.743	222.913	503.238
200	0.655	0.991	13.489	30.407	7.124	23.565	302.189	371.351
225	0.781	0.947	17.208	29.060	9.185	17.968	297.829	330.323
250	0.833	0.927	19.228	27.880	10.159	15.614	294.699	312.342
275	0.847	0.916	20.153	27.755	10.464	14.845	296.935	309.322
300	0.856	0.909	20.797	27.131	10.785	14.484	297.966	307.579
325	0.864	0.905	21.290	26.572	11.164	14.105	298.225	306.465
350	0.866	0.903	21.520	26.326	11.305	13.962	298.355	306.112
375			21.644	26.177	11.388	13.874	298.242	305.793
400			21.759	26.087	11.449	13.811	298.411	305.883
425			21.817	26.034	11.482	13.780	298.489	305.922
450			21.841	26.008	11.496	13.765	298.467	305.881
475			21.856	25.994	11.501	13.754	298.416	305.824
500			21.991	26.113	11.461	13.684	298.689	306.074
700			22.237	26.336	11.324	13.436	297.002	304.322

Table B11. The estimated parameters for case 6 (scenario 2)

S	D	C	$\hat{R}$	$\hat{\alpha}_3$	$\hat{\beta}$	$\hat{\mu}$	$V(\hat{R})$	$V(\hat{\alpha}_3)$	$V(\hat{\beta})$	$V(\hat{\mu})$	$V(\hat{\alpha})$
200	3.89	746.11	0.940	46.993	6.453	303.256	6.657E-03	2.965E+02	7.912E+00	4.923E+02	2.965E+02
225	24.30	725.70	0.946	47.618	6.333	301.584	3.273E-03	2.074E+02	4.730E+00	2.138E+02	2.074E+02
250	89.90	660.10	0.946	47.774	6.316	301.717	6.588E-04	4.663E+01	9.719E-01	2.707E+01	4.663E+01
275	218.39	531.61	0.947	47.863	6.295	301.285	2.038E-04	2.105E+01	4.069E-01	7.619E+00	2.105E+01
300	387.69	362.31	0.947	47.929	6.287	301.342	8.706E-05	1.179E+01	2.156E-01	3.732E+00	1.179E+01
325	545.82	204.18	0.949	47.907	6.278	300.740	5.122E-05	8.368E+00	1.477E-01	2.810E+00	8.368E+00
350	654.92	95.08	0.948	47.940	6.279	301.002	3.892E-05	6.684E+00	1.157E-01	2.579E+00	6.684E+00
375	712.65	37.35	**0.950**	47.927	6.280	300.978		6.081E+00	1.044E-01	2.508E+00	6.081E+00
400	737.80	12.20	**0.950**	47.932	6.266	300.323		5.799E+00	9.922E-02	2.490E+00	5.799E+00
425	746.60	3.40	**0.950**	47.939	6.255	299.865		5.672E+00	9.693E-02	2.485E+00	5.672E+00
450	749.11	0.89	**0.950**	47.929	6.242	299.168		5.624E+00	9.175E-02	2.458E+00	5.624E+00
475	749.84	0.16	**0.950**	48.104	6.233	299.831		5.495E+00	8.715E-02	2.564E+00	5.495E+00
500	749.98	0.02	**0.950**	48.169	6.233	300.217		5.478E+00	8.097E-02	2.612E+00	5.478E+00
700	750.00	0.00	**0.950**	48.158	6.245	300.725		5.458E+00	8.053E-02	2.610E+00	5.458E+00

Table B12. 90% confidence intervals for estimated parameters from case 6 (scenario 2)

s	90% confidence interval for $\hat{R}$		90% confidence interval for $\hat{\alpha}_3$		90% confidence interval for $\hat{\beta}$		90% confidence interval for $\hat{\mu}$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
200	0.806	1.000	18.666	75.319	1.826	11.080	266.757	339.756
225	0.852	1.000	23.929	71.307	2.756	9.911	277.531	325.637
250	0.904	0.988	36.541	59.007	4.694	7.937	293.159	310.275
275	0.924	0.971	40.316	55.410	5.245	7.344	296.745	305.826
300	0.932	0.963	42.281	53.578	5.523	7.051	298.164	304.520
325	0.937	0.961	43.149	52.666	5.645	6.910	297.983	303.498
350	0.938	0.958	43.687	52.193	5.719	6.838	298.360	303.644
375			43.871	51.984	5.748	6.811	298.373	303.583
400			43.971	51.894	5.747	6.784	297.728	302.919
425			44.021	51.857	5.743	6.767	297.271	302.458
450			44.028	51.830	5.744	6.740	296.589	301.747
475			44.248	51.960	5.747	6.719	297.197	302.465
500			44.319	52.019	5.764	6.701	297.558	302.875
700			44.314	52.001	5.778	6.711	298.067	303.383

APPENDIX C. THE MAXIMUM LIKELIHOOD PROPERTIES

We review some useful methods, properties and theorems related to MLE that we have discussed in our analysis. More detail can be found from Serfling (1980) and Lawless (1982)

Defined $\theta \equiv (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p)^T$ and P denotes the number of estimated parameters

Let $\hat{\theta}_n$ is an MLE for θ

$Y_1, Y_2, \ldots, Y_n \sim$ iid with density or mass functions that depend on θ; $f(y_i;\theta)$

1. Invariance:

   If $\hat{\theta}_n \equiv (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p)^T_n$ is MLE for θ, and if $g(.)$ is a real-valued function then $g(\hat{\theta})$ is an MLE $g(\theta)$

2. Asymptotic normality:

   MLE $\hat{\theta}_n$ is $AN\left(\theta, \frac{1}{nI(\theta)}\right)$

3. Fisher Information (single variable)

$$I(\theta) \equiv \left[E\left\{ \frac{\partial}{\partial\theta_i} \log f(y_i;\theta) \frac{\partial}{\partial\theta_j} f(y_i;\theta) \right\} \right]_{p\times p}, i, j = 1,2,\ldots,p$$

$$= \left[-E\left\{ \frac{\partial^2}{\partial\theta_i \partial\theta_j} \log f(y_i;\theta) \right\} \right]_{p\times p}$$

   Then, the total information is;

$$I_{tot}(\theta) = \sum_{i=1}^{n} -\left(E\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log f(y_i;\theta) \right] \right)_{p\times p} = nI(\theta)$$

4. Efficiency for estimation of a scalar parameter

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{nI(\theta)}\left[\frac{\partial}{\partial\theta}E(\hat{\theta})\right]^2$$

$\hat{\theta}$ is called unbiased for θ if $E(\hat{\theta}) = \theta$

Let $V(\hat{\theta}) \geq \frac{1}{nI(\theta)}$

If $\hat{\theta}$ is unbiased and $V(\hat{\theta}) = \frac{1}{nI(\theta)}$ then $\hat{\theta}$ is Minimum Variance Unbiased (MVU)

5. Linear transformations

If $g(\theta) = a\theta + b$, when θ is $AN(\mu, \sigma^2)$.

Then

$$g(\theta) \text{ is } AN(a\mu + b, a^2\sigma^2)$$

6. Delta method:

By using rule of linear transformation of normal

If $\hat{\theta}_n$ is $AN\left(\theta, \frac{V(\theta)}{n}\right)$ and $g(.)$ is a real value function

Such that,

$$g'(\theta) = \frac{d}{d\theta}g(\theta) < \infty$$

$$g'(\theta) \neq 0$$

Then

$$g(\hat{\theta}_n) \text{ is } AN\left(g(\theta), [g'(\theta)]^2 \frac{V(\theta)}{n}\right)$$

7. Approximate Interval:

An $(1-\varphi)100\%$ approximate interval for θ is

$$\left(\hat{\theta}_n - Z_{1-\frac{\varphi}{2}}\left[\frac{1}{nI(\theta)}\right]^{1/2}, \hat{\theta}_n - Z_{1-\frac{\varphi}{2}}\left[\frac{1}{nI(\theta)}\right]^{1/2}\right)$$

APPENDIX D. THE GAMMA AND ITS RELATED FUNCTIONS

We review some useful functions related with Gamma that we have discussed in our analysis using notation from Lawless (1982).

Likelihood function for Gamma with incomplete samples.

$$l(\theta)=\prod_{i\in D} f_i(y_i|\theta)\prod_{i\in C} S_i(y_i)$$

Where, $l(\theta)$ is a likelihood function with parameter θ.

$f_i(y_i|\theta)$ is a probability density function or probability mass function of observed data

$S_i(y_i)$ is survivor function for the censored data, $(S_i(y_i)=1-F_i(y_i|\theta))$

where, y_i is the survival time

APPENDIX E. MATHEMATICA PROGRAM

```
Needs["Statistics`ContinuousDistributions`"]
Do[
  dist1 = GammaDistribution[4, 25];
  dist2 = GammaDistribution[4, 25];
  distlength = 750;
  R1 = RandomArray[dist1, distlength];
  R2 = RandomArray[dist2, distlength];
  xx = R1 + R2;
  censortime = {150, 175, 200, 225, 250,
    275, 300, 325, 350, 375, 400, 425, 450, 475, 500, 2000};
  Array[ralpa1; rbeta1; lxA1; lcA1; alpaInfo; betaInfo; meanInfo;
   varianceInfo; probA1; probInfo; alpaInfonew;
   meanInfonew; varianceInfonew; probInfonew, {18, 100}];


  Do[c = censortime[[j]];
   x1 = Select[xx, # <= c &];
   c1 = Select[xx, # > c &];
   Lx1 = Length[x1];
   Lc1 = Length[c1];
   t1 = (Sum[x1[[i]], {i, 1, Lx1}]) / Lx1;
   t2 = (Product[x1[[i]], {i, 1, Lx1}]) ^ (1 / Lx1);
   r = Lx1;
   n = distlength;
   β = 6.25;
   sα := 3;
   sβ := 35;
   If[c > 150, sα = alpa1, sα = 3];

   logL1 = -r α Log[β] - r Log[Gamma[ α]] +
     r (α - 1) Log[t2 ] - (r t1 / β) + (Lc1) Log[Gamma[α, c / β] / Gamma[α]];
   logLInfo = -r γ Log[λ] - r Log[Gamma[ γ]] + r ( γ - 1) Log[t2 ] -
     (r t1 / λ) + (Lc1) Log[Gamma[ γ, c / λ] / Gamma[ γ]];

   SDα = -D[logLInfo, { γ, 2}];
   SDβ = -D[logLInfo, {λ, 2}];
   PDαβ = -D[logLInfo, γ, λ];

   p1 = Integrate[PDF[GammaDistribution[θ, ρ], t], {t, 0, c}];
   p2 = D[p1, θ];
   p3 = D[p1, ρ];

   alpa1 = α /. FindMinimum[-logL1, {α, sα},
       MaxIterations → 30000, AccuracyGoal → 2, WorkingPrecision → 2][[2]];
   beta1 = β;
```

```
γ = alpal;
λ = betal;
θ = alpal;
ρ = betal;
Clear[sα, sβ];
ralpal[j, k] = alpal;
rbetal[j, k] = betal;

lxAl[j, k] = Lxl;
lcAl[j, k] = Lcl;
sα := alpal;
sβ := betal;

ϕ = alpal;
σ = betal;
x = c / σ;

Q1 = Integrate[e^-t t^(ϕ-1) Log[t], {t, x, Infinity}] // N;
Q2 = Integrate[e^-t t^(ϕ-1) Log[t]^2, {t, x, Infinity}] // N;
Q3 = Gamma[ϕ, x] / Gamma[ϕ];
Q4 = (e^-x c^ϕ σ^(-ϕ-1)) / Gamma[ϕ];
Q5 = (e^-x c^ϕ σ^(-ϕ-2)) / Gamma[ϕ] * (x - (ϕ + 1));
Q6 = Q1 / Gamma[ϕ] - (Pg1 * Q3);
Q7 = Q4 * (Log[c] - Log[σ] - Pg1);
Q8 = Integrate[e^(-t/σ) t^(ϕ-1) Log[t], {t, 0, c}] // N;
Q9 = Integrate[e^(-t/σ) t^(ϕ-1) , {t, 0, c}] // N;
Q10 = Integrate[e^(-t/σ) , {t, 0, c}] // N;
Q11 = Integrate[e^(-t/σ) t^ϕ , {t, 0, c}] // N;
Q12 = Q2 / Gamma[ϕ] - ((Pg1 * Q1) / Gamma[ϕ]) - (PolyGamma[1, ϕ] * Q3) - (Pg1 * Q6);

SDLα2UC = r * PolyGamma[1, ϕ];
SDLα2C1 = - (n - r) * (Q12 / Q3 - Q6^2 / Q3^2);
SDLα2 = SDLα2UC + SDLα2C1;

SDLβ2UC = (2 * t1 * r) / σ^3 - ((r * ϕ) / σ^2);
SDLβ2C1 = - (n - r) * (Q5 / Q3 - Q4^2 / Q3^2);
SDLβ2 = SDLβ2UC + SDLβ2C1;
```

$$PDL\alpha\beta UC = \frac{r}{\sigma};$$

$$PDL\alpha\beta C1 = -(n-r) * \left(\frac{Q7}{Q3} - \left(\frac{Q6 * Q4}{Q3^2}\right)\right);$$

$$SDL\alpha\beta = PDL\alpha\beta UC + PDL\alpha\beta C1;$$

$$PDP\alpha = \frac{e^{-\frac{t}{\sigma}}\, t^{-1+\phi}\, \sigma^{-\phi}\, \mathrm{Log}[t]}{\mathrm{Gamma}[\phi]} - \frac{e^{-\frac{t}{\sigma}}\, t^{-1+\phi}\, \sigma^{-\phi}\, \mathrm{Log}[\sigma]}{\mathrm{Gamma}[\phi]} - \frac{e^{-\frac{t}{\sigma}}\, t^{-1+\phi}\, \sigma^{-\phi}\, \mathrm{PolyGamma}[0, \phi]}{\mathrm{Gamma}[\phi]};$$

$$PDP\beta = \frac{e^{-\frac{t}{\sigma}}\, t^{\phi}\, \sigma^{-2-\phi}}{\mathrm{Gamma}[\phi]} - \frac{e^{-\frac{t}{\sigma}}\, t^{-1+\phi}\, \sigma^{-1-\phi}\, \phi}{\mathrm{Gamma}[\phi]};$$

```
DPDPα = Integrate[PDPα, {t, 0, c}] // N;
DPDPβ = Integrate[PDPβ, {t, 0, c}] // N;

Totalα = SDLα2;
Totalβ = SDLβ2;
Totalαβ = SDLαβ;

Info = {{Totalα, Totalαβ}, {Totalαβ, Totalβ}};
InfoInverse = Inverse[Info];

alpaInfo[j, k] = InfoInverse[[1, 1]];
betaInfo[j, k] = InfoInverse[[2, 2]];
alpaInfonew[j, k] = 1 / Totalα;

a = {{λ, γ}, {λ^2, 2 γ λ}};
i11 = InfoInverse[[1, 1]];
i22 = InfoInverse[[2, 2]];
i12 = InfoInverse[[1, 2]];
b = {{i11, i12}, {i12, i22}};
ca = Transpose[a];
d = a.b.ca;
meanInfo[j, k] = d[[1, 1]];
varianceInfo[j, k] = d[[2, 2]];

meanInfonew[j, k] = ((σ^2) / Totalα);
varianceInfonew[j, k] = ((σ^4) / Totalα);


y2 = p2 // N;
y3 = p3 // N;
m1 = {{DPDPα, DPDPβ}};
m2 = Transpose[m1];
m3 = m1.b.m2 // N;

probA1[j, k] = p1;
probInfo[j, k] = m3;
```

```
probInfonew[j, k] = ((DPDPα) ^2) / Totalα;

Clear[γ, λ, θ, ρ, sα, sβ]

, {j, 1, Length[censortime]}];
Print["End of Loop"[k]], {k, 1, 5}] // Timing
```

REFERENCES

Brockman, T. (1999). "21 warehousing trends in the 21st century." *IIE Solutions* 37(7): 36.

Caldwell, B. (1999). "Reverse logistics--untapped opportunities exist in returned products, a side of logistics few business have thought about-until now. (company operations)." *Information week* April 1999: p 48.

Cox, D. R. and Oakes, D. (1984). Analysis of survival data, Chapman and Hall.

de Brito, M. P. and Dekker, R. (2002). Reverse logistics-a framework, Econometric Institute Report EI 2002-38, Erasmus University Rotterdam, The Netherlands

de Brito, M. P. and Dekker, R. (2001). Modeling product returns in inventory control exploring the validity of general assumptions, Econometric Institute Report EI 2001-27, Erasmus University Rotterdam, The Netherlands

Degher, A. (2002). "HP's worldwide take back and recycling programs: lessons on improving program implementation." *IEEE International Symposium on Electronics and the Environment*, 2002, p 224-227

Fleischmann, M., Bloemhof-Ruwaard J.M., Dekker, R., Van der Laan, E., van Nunen, Jo. A.E.E., Van Wassenhove, L.N.(1997). "Quantitative models for reverse logistics: a review." *European journal of operational research* 103: 1-17.

Gungor, A. and Surendra, G. (1999). "Issues in environmentally conscious manufacturing and product recovery: a survey." *Computers & industrial engineering* 36: 811-853.

Goggin, K. and Jim, B. (2000). "Towards a taxonomy of resource recovery from end-of-life products." *Computers in industry* 42: 177-191.

Goh, T. N. and Varaprasad, N. (1983). "A statistical methodology for the analysis of the life-cycle of reusable containers." *IIE Transactions* 18: 42-47.

Gooley, T. B. (1998). "Reverse logistics: five steps to success." *Logistics management & distribution report* 37(6): 49.

Grenchus, E., Johnson, S., McDonnell, D. (2001). "Improving environmental performance through reverse logistics at IBM." *IEEE International Symposium on Electronics and the Environment*, 2001,p236-240.

Grenchus, E., Keene, R., Luce, R., Nobs, C. (2002). "Composition and value of returned industrial information technology equipment revisited". *IEEE International Symposium on Electronics and the Environment*, 2002, p 214-217

Grenchus, E., Keene, R., R., Nobs, C (2000). "Composition and value of returned consumer and industrial information technology equipment." *IEEE International Symposium on Electronics and the Environment*, 2000, p 324-329

Guide, V. D. R. Jr. (2000). "Production planning and control for remanufacturing: industry practice and research needs." *Journal of operations management* 18: 467-483.

Kelle, P. and Silver, E. A. (1989a). "Purchasing policy of new containers considering the random returns of previously issued containers." *IIE Transactions* 21(4).

Kelle, P. and Silver, E. A. (1989b). "Forecasting the returns of reusable containers." *Journal of operations management* 8(1): 17-35.

Kokkinaki, A. I., Dekker, R., van Nunen, J., Pappis, C. (2000). "An exploratory study on electronic commerce for reverse logistics." *Supply chain forum* (1): 10.

Krupp, J. (1992). "Core obsolescence forecasting in remanufacturing." *Production and Inventory Management Journal* 33(2): 12-17.

Lawless, J. F. (1982). *Statistical models and methods for lifetime data*, Wiley New York.

Lee, J., McShane, H., Kozlowski, W. (2002). Critical issues in establishing a viable supply chain/verse logistic management program. *IEEE International symposium on electronics and the environment*, p150-156

Lund, R. I. (1984). "Remanufacturing." *Technology review* 87(2): 18-23.

Majumder, P. and Groenevelt, H. (2001). "Competition in remanufacturing." *Production and operations management* 10(2): 125.

Muller, A. (1999). "Using end-of-life estimates to perform design for environment investment analysis." *IEEE international symposium on electronics and environment,* 1999, p320-324.

Product life cycle data model, American standard ANSI/EIA-724, Sept.19, 1997.

Punt, A.E., Hilborn, R. Bayesian stock assessment methods in fisheries-user's manual, Retrieved April 18, 2003, from http://www.foa.org/DOCREP/005/Y1958E/y1958e00.htm

National Institute of Standards and Technology, Retrieved March 30, 2003, from http://www.itl.nist.gov/div898/software/dataplot.html/refman1/ch2/relsd.pdf.

Richardson, H. L. (2001). "Logistics in reverse." *Industry week* 250: 37-40.

Ritchey, J., Mahmoodi, F., Frascatore, M., Zander, A.. (2001). "Assessing the technical and economic feasibility of remanufacturing." *Proceedings of the twelfth annual conference of the production and operations management society*, Orlando, Florida.

Rogers, D. S. and Tibben-Lembke, R. S. (1999). *Going backwards: Reverse logistics trends and practices*, University of Nevada, Reno.

Saar, S. and Thomas, V. (2002). "Advanced product tags for recycling." Proceedings from the 2002 *IEEE international symposium on electronics and the environment,* San Francisco, California.

Serfling, R.J. (1980). *Approximation theorems of mathematical statistics*, Wiley New York

Srivastava, R. and Guide, V. D. R. Jr. (1995). "Forecasting for parts recovery in a remanufacturing environment." *Decision science institute*: 1206-1208.

The Reverse Logistics Executive Council, Retrieved November 18, 2002, from http://www.rlec.org.

Toktay, B., Wein, L., Zenios, S.A. (2000). "Forecasting product returns" *Management science* 46, p1412-1426.

Wolfram, S. (1999). *The Mathematica book*, 4th ed., Cambridge University Press, New York.